American Journal of Sciences and Engineering Research www.iarjournals.com

American Journal of Sciences and Engineering Research

E-ISSN -2348 – 703X, Volume 8, Issue 2, 2025

Fixed Point Theorems of Multiplicative *G*-Metric Spaces with Application of Traveling Salesman Problem

Muhammad Bilal

School of Mathematics and Statistics, Yunnan University, Kunming, 650106 China

Abstract

Multiplicative G-metric space is an extension of the well-known multiplicative metric space with the structure of three arguments contains ordinary multiplication. We in this paper establish some fixed point theorems of multiplicative G-metric spaces under some contraction conditions. These results are also given for the closed balls of multiplicative G-metric spaces. Initially we proved fixed point theorem by taking single map and then we extended our idea to two maps. At the end of this paper, we have given an application of Multiplicative G-metric space in traveling salesman problem for finding shortest path between different cities. We gave C++ Code for traveling salesman problem to find shortest path and shortest distance. Our technique for finding shortest path and shortest distance for traveling salesman is more robust. We also gave example to support our theorem.

Keywords: *Fixed point, Multiplicative G-metric space, Contraction mappings, Double map, traveling salesman problem.*
2010 MSC: 47H10,54H25.

1. Introduction

Metric space known as the distance function was instigated by M. Frechet in 1906 [1]. Over two decades, S. Banach [3] generalized the concept, and then studied it systematically in 1920-1922 along with Hans Hahn and Eduard Elly, and gave various results, such as Hahn-Bannach theorem. Since then, many generalizations of a metric space model have been defined by many researchers. For example, S. Gahler in 1968 introduced the notion of 2-metric space [8, 9]. It was the first structure on three arguments before this structure, metric space was introduced which was the structure on two arguments. Approximately over two decades the study of 2-metric spaces was studied by mathematicians. In

 Received-18-03-2025, Accepted- 27-03-2025

1984, B.C. Dhage [22] an Indian mathematician introduced the new structure on three arguments named by *D*-metric space as the generalized structure of metric space. He wrote his PhD thesis on *D*-metric space. The difference between 2- metric space and *D*-metric space was that 2-metric represents the perimeter of triangle while *D*-metric represents the area of triangle. After this there was a spat of papers were published on this newly introduced structure *D*-metric space by Dhage, for details see [10, 11, 12, 13]. The study of this structure also been studied by many researchers over two decades. Then an Egyptian mathematician in 2003 with his Supervisor, Z. Mustafa and B. Sims [14] gave some remarks on the structure of *D*-metric space. They claimed that the structure of *D*- metric space is not the generalization of usual metric space in general. In 2003, after giving remarks on *D*-metric space, Z. Mustafa and B. Sims [15] in 2005 gave the more robust structure of generalized metric space named as *G*-metric space. After that, some papers on this new structure appears, see [16, 17, 18]. Subsequently, a new structure of metric space called multiplicative metric space was introduced by Bashirov et al in2008 [19]. In this structure, they introduced the contraction condition for multiplicative metric space which is quite different from the definition of contraction condition in usual metric space. They used multiplication in triangle inequality and also showed that the distance of two points is greater than or equal to 1. The structure of fixed point theory is most important in analysis. The first systematic way of finding fixed points of self mappings was initiated by S. Bannach in his famous contraction principle [3]. This principle is widely used in analysis of operators. Many researchers proved common fixed point theorems on many generalizations of fixed point theory. For example, A. Alrazi and J. Ahmad [3] proved *L*-fuzzy mappings and common fixed point theorems. A. Azam and I. Beg [4] proved common fixed points of fuzzy maps. A. Azam [5] also proved fuzzy fixed point of fuzzy mapping by taking rational inequality. Ozavasar and Ceikel [19] proved fixed point theorems on multiplicative metric space.

Recently, the structure of multiplicative *G*-metric spaces is presented by P. Nagpal et al. in 2016 [6] and generalized Bannach fixed point theorem in the setting of multiplicative *G*-metric space. They also extended the generalization for two maps in the settings of multiplicative *G*-metric space. And then M. Mazhar et al. [7] proved the rational type fixed point theorems on multiplicative *G*-metric spaces, they initially proved the fixed point theorems by taking single maps. Later on, they also proved fixed points by taking triplet maps.

In this paper, we also focus on the fixed point theory of multiplicative *G*-metric spaces, and give some fixed point theorems by taking structure of multiplicative *G*-metric spaces initially we proved fixed point theorem by taking single map and later on we extended our idea and we proved some theorems using double maps. We also gave robust example to support our theorem. At the end of this paper, we also give real life application of Multiplicative *G*-metric space in well known problem of graph theory which is travelling salesman problem. The main purpose of this problem is to find the shortest path for salesman who is selling his products to different cities and want to cover cities in a less time. We use C++ code for finding shortest path and shortest distance of this shortest path. Our results are more robust than other methods because other methods are time consuming. By using code we can get our required shortest path and shortest distance more faster than any other existing method.

2. Preliminaries

Now we list some definitions, notation and lemmas which are used in the next.

Definition 1. [6] Let X be a non-empty set. A function $d : X \times X \to R^+$ is said to be a multiplicative metric on X if for any $x, \ y, \ z \in X$ the following conditions hold:

(i) $d(x, y) \geq 1$ and $d(x, y) = 1 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$ (Symmetry);
(iii) $d(x, y) \leq d(x, z)d(z, y)$ (Triangle inequality).

The pair $(X, \ d)$ is called a multiplicative metric space.

Definition 2. [6] Let X be a non-empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following conditions:

(i) $G(x, y, z) = 1$ if and only if $x = y = z$;
(ii) $1 < G(x, x, y)$, $\forall x, y \in X$ with $x \neq y$;
(iii) $G(x, x, y) \leq G(x, y, z)$, $\forall x, y, z \in X, z \neq y$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry);
(v) $G(x, y, z) \leq G(x, a, a)G(a, y, z)$, $\forall x, y, z, a \in X$ (rectangle inequality);

The pair (X, G) is called a multiplicative G-metric space.

Here we give two examples for multiplicative G-metric space from [6].

Example 1. [6] Let R be the set of all real numbers. Define a function $G : R \times R \times R \to R^+$ by

$$G(x, y, z) = e^{|x-y|+|y-z|+|z-x|} \qquad \forall x, \ y, \ z \in R.$$

Then the pair (R, G) is a multiplicative G-metric space.

Example 2. [6] Let (X, d) be an usual multiplicative metric space and define $G : X \times X \times X \to R^+$ by

$$G(x, y, z) = d(x, y)d(y, z)d(x, z), \ for \ all \ x, \ y, \ z \in X.$$

Then (X, G) is a multiplicative G-metric space.

For a multiplicative G-metric space, there are some fundamental and important inequalities as below.

Proposition 1. [6] Let (X, G) be a multiplicative G-metric space. Then for any $x, y, z \in X$ and $a \in X$, the following conditions hold:

(i) $G(x, y, z) \leq G(x, x, y)G(x, x, z)$;
(ii) $G(x, x, y) \leq 2G(y, x, x)$;
(iii) $G(x, y, z) \leq G(x, a, z)G(a, y, z)$;
(iv) $G(x, y, z) \leq \frac{2}{3}\{G(x, y, a)G(x, a, z)G(a, y, z)\}$;
(v) $G(x, y, z) \leq G(x, a, a)G(y, a, a)G(z, a, a)$.

Definition 3. [6] Let (X, G) be a multiplicative G-metric space and $\{x_n\}$ be a sequence of points of X. The sequence $\{x_n\}$ is said to be multiplicative G-convergent to x if

$$\lim_{n,\ m\to\infty} G(x_n, x_m, x) \to 1,$$

that is, for every $\epsilon > 1$, there exist a number $n_0 \in N$ such that

$$G(x_n, x_m, x) < \epsilon.$$

Definition 4. [6] Let (X, G) be a multiplicative G-metric space. A sequence $\{x_n\}$ is called multiplicative G-Cauchy if for a given $\epsilon > 1$, there exist $n_0 \in N$ such that for all $m, n, l \geq n_0$,

$$G(x_m, x_n, x_l) \leq \epsilon,$$

that is, if

$$G(x_m, x_n, x_l) \to 1 \ as\ n, m, l \to \infty$$

for all $m, n \geq n_0$.

Definition 5. [6] Let (X, G) and (X', G') be two multiplicative G-metric spaces and $f : (X, G) \to (X', G')$ be a function. Then f is said to be a multiplicative G-continuous at a point $a \in X$ if for any $\epsilon > 1$, there exist $\delta > 1$ such that $G(a, x, y) < \delta$ implies $G'(fa, fx, fy) < \epsilon$, for $x, y \in X$. Furthermore, a function f is said to be multiplicative G-continuous on x if and only if it is multiplicative G-continuous at all $a \in X$.

Definition 6. [6] A multiplicative G-metric space (X, G) is said to be multiplicative G-complete if every multiplicative G-Cauchy sequence in (X, G) is multiplicative G-convergent in (X, G).

Definition 7. [6] Let (X, G) be a multiplicative G-metric space, any $x \in X$ and $\epsilon > 1$. A set

$$B_\epsilon(x_0) = \{y \in X : G(x_0, y, y) < \epsilon\}$$

is called a multiplicative $G-$open ball of radius ϵ and with center x_0. Similarly, the set

$$\overline{B_\epsilon(x_0)} = \{y \in X : G(x_0, y, y) \leq \epsilon\}$$

is called a multiplicative G-closed ball.

Lemma 1. *[6] A multiplicative G-Cauchy sequence is multiplicative G-bounded.*

Proof. Let (X, G) be a multiplicative G-metric space and $\{x_n\}$ be a multiplicative G-Cauchy sequence in it. From Definition 4, it implies that for $\epsilon = 2 > 1$ there exists a natural number n_0 such that $G(x_n, x_m, x_l) < 2$ for all $m, n \geq n_0$. Hence, if we set

$$M = \max\{2, G(x_1, x_{n_0}, x_{n_0}), \ldots, G(x_{n_0-1}, x_{n_0}, x_{n_0})\},$$

then it is clear that

$$G(x_n, x_{n_0}, x_{n_0}) < M, \text{ for all } n \in N.$$

Thus we have

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n_0}, x_{n_0})G(x_m, x_{n_0}, x_{n_0}) < M^2, \text{ for all } m, n \in N.$$

This implies that the sequence $\{x_n\}$ is multiplicative G-bounded. □

Definition 8. [6] Let (X, G) be a multiplicative G-metric space. A mapping $f : X \to X$ is said to be a multiplicative G-contraction if there exist $\lambda \in [0, 1)$ such that

$$G(fx, fy, fz) \leq G(x, y, z)^\lambda, \forall\ x, y, z \in X.$$

Definition 9. [7] Let $f : X \to X$ be a mapping. The point $x \in X$ is called a fixed point if $f(x) = x$.

3. Main results

In this section, we establish fixed point theorems by taking the contraction conditions of Multiplicative G-Metric spaces with single maps, and extended this to double mapping in the setting of multiplicative G-Metric Spaces.

Theorem 1. *Let (X, G) be a complete multiplicative G-metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point if*

$$G(fx, fy, fz) \leq (G(x, y, z))^\lambda, \text{ for all } x, y, z \in X \text{ and } \lambda \in [0, 1).$$

Proof. Let x_0 be any arbitrary point of X and define the sequence $\{x_n\}$ in X by $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$.

Assume that $x_n \neq x_{n+1}$, from $x_{n+1} = f(x_n)$ we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\
&\leq (G(x_{n-1}, x_n, x_n))^\lambda \\
&= (G(fx_{n-2}, fx_{n-1}, fx_{n-1}))^\lambda \\
&\leq (G(x_{n-2}, x_{n-1}, x_{n-1}))^{\lambda^2} \\
&\vdots \\
&\leq (G(x_0, x_1, x_1))^{\lambda^n}.
\end{aligned}$$

By using multiplicative triangle inequality, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\le (G(x_n, x_{n+1}, x_{n+1}))^{\lambda}(G(x_{n+1}, x_{n+2}, x_{n+2}))^{\lambda}\ldots \\ &\quad (G(x_{m-1}, x_m, x_m))^{\lambda} \\ &\le (G(x_0, x_1, x_1))^{\lambda^n}(G(x_0, x_1, x_1))^{\lambda^{n+1}}(G(x_0, x_1, x_1))^{\lambda^{m-1}} \\ &= (G(x_0, x_1, x_1))^{\lambda^n+\lambda^{n+1}+\ldots+\lambda^{m-1}} \\ &= (G(x_0, x_1, x_1))^{\lambda^n(1+\lambda+\lambda^2+\ldots+\lambda^{m-n-1})} \\ &= (G(x_0, x_1, x_1))^{\lambda^n, \frac{1-\lambda^{n-m}}{1-\lambda}}. \end{aligned}$$

As $\lambda < 1$ and $m, n \to \infty$, so $1 - \lambda^{n-m} < 1$ and

$$G(x_n, x_{n+1}, x_{n+1}) \le (G(x_0, x_1, x_1))^{\lambda^n}.$$

This implies $G(x_n, x_m, x_m) \to 1$ as $m, n \to \infty$. i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$, and thus $\lim_{m,n\to\infty}(G(X_0, X_1, X_1))^{\lambda^n\left(\frac{1-\lambda^{n-m}}{1-\lambda}\right)} = 1$. So the Sequence $\{x_n\}$ is a Cauchy Sequence.

Existence of the fixed point: By the completeness of G, there exist a point $u \in X$ such that $\{x_n\}$ is a multiplicative G-convergent to u. Since

$$G(fx_n, fu, fu) \le G(x_{n-1}, u, u)$$

and $x_{n-1} \to u$ as $n \to \infty$, then $G(x_n, fu, fu) \le (G(u, u, u))^{\lambda^n}$. Hence $f(u) = u$, that is, u is a fixed point.

Uniqueness of the fixed point: Suppose that v is another fixed point of f, from $f(v) = v$ we have

$$G(u, v, v) = G(fu, fv, fv) \le (G(u, v, v))^{\lambda}.$$

This shows that $v = u$. The proof is completed. □

Next we give an example to show Theorem 1.

Example 3. Let R be the set of all real numbers. Consider the function

$$G(x, y, z) = e^{|x-y|+|y-z|+|z-x|} \qquad \forall x, y, z \in R$$

in Example 1. Let $f : X \to X$ be defined by

$$G(fx, fy, fz) \le (G(x, y, z))^{\lambda}, \text{ for all } x, y, z \in X.$$

Then, f has a unique fixed point by using theorem 1.

Corollary 1. Let (X, G) be a complete multiplicative G-metric space and $f : X \to X$ be a contraction map. Then f has a fixed point if

$$G(fx, fy, fz) \le (G(x, y, z))^{\frac{1}{2}} \text{ for all } x, y, z \in X \text{ and } \lambda \in [0, 1/2), \qquad (1)$$

Example 4. Let $X = \{0, \frac{1}{2}, 1\}$, Define, $G: X \times X \times X \to R$ by $G(0, 1, 1) = 2 = G(1, 0, 0)$

$G(0, \frac{1}{2}, \frac{1}{2}) = 1 = G(\frac{1}{2}, 0, 0)$

$G(\frac{1}{2}, 1, 1) = \frac{3}{2} = G(1, \frac{1}{2}, \frac{1}{2})$

$G(0, \frac{1}{2}, 1) = \frac{3}{2}$

$G(x, x, x) = 1 for all x \in X$

Let $f : X \to X$ be defined by

$f(0) = 0, f(\frac{1}{2}) = \frac{1}{2}, f(1) = 0$

$G(f(0), f(\frac{1}{2}), f(\frac{1}{2})) = G(0, \frac{1}{2}, \frac{1}{2}) = 1$

$G(f(0), f(1), f(1)) = G(0, 0, 0) = 1$

$G(f(\frac{1}{2}), f(1), f(1)) = G(\frac{1}{2}, 0, 0) = 1$

$G(f(0), f(\frac{1}{2}), f(1)) = G(0, \frac{1}{2}, 0) = 1$

Now,

$$1 = G(f(0), f(\frac{1}{2}), f(\frac{1}{2})) = G(0, \frac{1}{2}, \frac{1}{2}) \tag{2}$$

Applying eq. (2) in eq. (1), we get

$= G(f(0), f(\frac{1}{2}), f(\frac{1}{2})) \leq G(0, \frac{1}{2}, \frac{1}{2})^{\frac{1}{2}}$

$1 = G(f(0), f(\frac{1}{2}), f(\frac{1}{2})) \leq (1)^{\frac{1}{2}}$

$1 = 1$

Now,

$$1 = G(f(0), f(1), f(1) = G(0, 0, 0) \tag{3}$$

Applying Eq. (3) in Eq. (1), we get,

$1 = G(f(0), f(1), f(1) \leq G(0, 0, 0)^{\frac{1}{2}}$

$1 = G(f(0), f(1), f(1) \leq (1)^{\frac{1}{2}}$

$1 = 1$

Again, consider,

$$1 = G(f(\frac{1}{2}), f(1), f(1)) = G(\frac{1}{2}, 0, 0) \tag{4}$$

Applying Eq. (4) in Eq. (1), we get

$1 = G(f(\frac{1}{2}), f(1), f(1)) \leq G(\frac{1}{2}, 0, 0)^{\frac{1}{2}}$

$1 = G(f(\frac{1}{2}), f(1), f(1)) \leq (1)^{\frac{1}{2}}$

$1 = 1$

So, f has two fixed point $0, \frac{1}{2}$

Corollary 2. Let (X, G) be a complete multiplicative G-metric space and $f : X \to X$ be a contraction map. Then f has a unique fixed point if

$$G(fx, fy, fz) \leq (G(x, y, z))^{\lambda} \text{ for all } x, y, z \in \overline{B}(x_o, r) \text{ and } \lambda \in [0, 1/2),$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda}.$$

Theorem 2. *Let f and g be two maps on a complete multiplicative G-metric space X, f, g : X → X and x_0 be any arbitrary point in X. Suppose that there exists $\lambda \in [0, 1)$ such that*

$$G(fx, gy, gy) \leq (G(x, y, y))^{\lambda}, \text{ for any } x, y \in X.$$

Then there exist a unique common fixed point of f and g in X.

Proof. Let x_0 be any given point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = f(x_{2n}) \text{ and } x_{2n+2} = g(x_{2n+1}),$$

Now,

$$\begin{aligned} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= G(f(x_{2n}), g(X_{2n+1}), g(X_{2n+1})) \\ &\leq G(x_{2n}, x_{2n+1}, x_{2n+1})^{\lambda} \\ &\leq G(f(x_{2n-1}), g(x_{2n}), g(x_{2n}))^{\lambda} \\ &\leq G(x_{2n-1}, x_{2n}, x_{2n})^{\lambda^2} \\ &\quad . \\ &\leq G(x_0, x_1, x_1)^{\lambda^{n+1}}. \end{aligned}$$

By using triangle inequality, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq (G(x_n, x_{n+1}, x_{n+1})).(G(x_{n+1}, x_{n+2}, x_{n+2}))... \\ &\quad (G(x_{m-1}, x_m, x_m)) \\ &\leq (G(fx_0, gx_1, gx_1))^{\lambda^n}.(G(fx_0, gx_1, gx_1))^{\lambda^{n+1}}... \\ &\quad (G(fx_0, gx_1, gx_1))^{\lambda^{m-1}} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n+\lambda^{n+1}+...+\lambda^{m-1}} \\ &= (G(x_0, x_1, x_1))^{\lambda^n+\lambda^{n+1}+...+\lambda^{m-1}} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n(1+\lambda+\lambda^2+...+\lambda^{m-n-1})} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n(\frac{1-\lambda^{n-m}}{1-\lambda})}. \end{aligned}$$

As $\lambda < 1$ and $m, n \to \infty$, so $1 - \lambda^{n-m} < 1$, then

$$G(x_n, x_m, x_m) \leq (G(x_0, x_1, x_1))^{\lambda^n}.$$

This implies that $G(x_n, x_m, x_m) \to 1$ as $m, n \to \infty$, i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$. Since $\lim_{m,n\to\infty}(G(x_n, x_m, x_m))^{\lambda^n(\frac{1-\lambda^{m-n}}{1-\lambda})} = 1$, the sequence $\{x_n\}$ is a Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $u \in X$ such that $\lim_{m,n\to\infty} x_n = u$. So

$$G(x_n, g(u), g(u)) = G(f(x_{n-1}), g(u), g(u)) \leq G(x_{n-1}, u, u)^{\lambda}.$$

As $u \to \infty$, So $x_{n-1} \to u$, then $G(x_n, g(u), g(u)) = G(u, u, u)^{\lambda}$. Thus f and g have a common fixed point u.

Uniqueness of the fixed point: Suppose that z is another fixed point such that

$$G(u, z, z) = G(fu, gz, gz) \leq G(u, z, z)^{\lambda},$$

which implies that $u = z$. So f and g have a common fixed point u. The proof is completed. □

Corollary 3. Let f and g be two maps on a complete multiplicative G-metric space X, $f, g : X \to X$ and x_0 be any arbitrary point in X. Suppose that $\exists\lambda \in [0, 1)$ such that

$$G(fx, gy, gy) \leq (G(x, y, y))^{\lambda}, \text{ for any } x, y \in \overline{B}(x_o, r),$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda},$$

then there exist a unique common fixed point of f and g in X.

Theorem 3. *Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f : X \to X$ satisfies the contraction condition*

$$G(fx, fy, fy) \leq (G(fx, x, x)G(fy, y, y))^{\lambda}$$

for all $X, y \in X$ where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f has a unique fixed point in X, and for any $x_0 \in X$ iterative sequence $(f^n x)$ converges to the fixed point.

Proof. Let x_0 ba any given point in X. Define an iterative sequence $\{x_n\}$ in X

$$\begin{aligned} x_1 &= f(x_0), \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0), \\ &\cdot \\ x_{n+1} &= f^n(x_0), \ n = 1, 2, ... \end{aligned}$$

Thus we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq (G(fx_{n-1}, x_{n-1}, x_{n-1})G(fx_n, x_n, x_n))^{\lambda} \\ &= (G(x_n, x_{n-1}, x_{n-1})G(x_{n+1}, x_n, x_n))^{\lambda}. \end{aligned}$$

As

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &\leq \{G(x_n, x_{n-1}, x_{n-1})\}^{\frac{\lambda}{1-\lambda}} \\ &= \{G(x_n, x_{n-1}, x_{n-1})\}^h, \end{aligned}$$

where $h = \frac{\lambda}{1-\lambda}$, we have for $n > m$,

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n-1}, x_{n-1})G(x_{n-1}, x_{n-2}, x_{n-2}) \cdots G(x_{m+1}, x_m, x_m)$$

$$
\begin{aligned}
&\le G(x_1, x_0, x_0)^{h^{n-1}} G(x_1, x_0, x_0)^{h^{n-2}} \cdots G(x_1, x_0, x_0)^{h^m} \\
&\le G(x_1, x_0, x_0)^{h^{n-1}+h^{n-2}+\ldots+h^m} \\
&\le G(x_1, x_0, x_0)^{\frac{h^m}{1-h}}.
\end{aligned}
$$

This implies that $G(x_n, x_m, x_m) \to 1$ as $n, m \to \infty$, i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$. Since $\lim_{n,m\to\infty}(G(x_0, x_1, x_1))^{\frac{h^m}{1-h}} = 1$, the sequence $\{x_n\}$ is Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $u \in X$ such that $x_n \to u$ $(n \to \infty)$. Since

$$
\begin{aligned}
G(fu, u, u) &\le (G(fx_n, fu, fu)G(fu_n, u, u))^{\lambda} \\
&\le (G(fx_n, x_n, x_n)G(fu, u, u))^{\lambda}\, G(x_{n+1}, u, u),
\end{aligned}
$$

we have

$$G(fu, u, u) \le \{G(fx_n, x_n, X_n)^{\lambda} \;\; G(x_{n+1}, u, u)\}^{\frac{1}{1-\lambda}} \to 1 \ (n \to \infty).$$

Hence

$$G(fu, u, u) = 1.$$

This implies that $fu = u$, and thus u is a fixed point of f.

Uniqueness of the fixed point: Suppose that v is another fixed point such that

$$G(u, v, v) = G(fu, fv, fv) \le \{G(fu, u, u).G(fv, v, v)\}^{\lambda}$$

which implies that $u = v$, and thus the fixed point is unique. The proof is completed. □

Corollary 4. Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f : X \to X$ satisfies the contraction condition

$$G(fx, fy, fy) \le \{G(fx, x, x).G(fy, y, y)\}^{\lambda}, \;\; for \;\; all \;\; x, y \in \bar{B}(x_0, r),$$

and

$$G(x_0, f(x_0), f(x_0)) \le r^{1-\lambda},$$

where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f has a unique fixed point in X and for any $x \in X$ iterative sequence $(f^n x)$ converges to the fixed point.

Theorem 4. *Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f : X \to X$ satisfies the contraction condition*

$$G(fx, fy, fy) \le (G(fx, y, y)G(fy, x, x))^{\lambda}$$

for all $x, y \in X$ where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f has a unique fixed point in X, and for any $x \in X$ iterative sequence $(f^n x)$ converges to the fixed point.

Proof. Let x_0 ba any given point in X. Define an iterative sequence $\{x_n\}$ in X such that

$$\begin{aligned} x_1 &= f(x_0), \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0), \\ &\quad . \\ x_{n+1} &= f^n(x_0), \ n = 1, 2, ... \end{aligned}$$

Thus we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \{G(fx_{n-1}, x_n, x_n).G(fx_n, x_{n-1}, x_{n-1})\}^\lambda \\ &\leq \{G(fx_{n-1}, x_n, x_n).G(fx_n, x_{n-1}, x_{n-1})\}^\lambda \\ &\leq \{G(x_n, x_n, x_n).G(x_{n+1}, x_{n-1}, x_{n-1})\}^\lambda. \end{aligned}$$

As

$$\begin{aligned} G(x_{n+1}, x_n, x_n) &\leq \{G(x_n, x_{n-1}, x_{n-1})\}^{\frac{\lambda}{1-\lambda}} \\ &= \{G(x_n, x_{n-1}, x_{n-1})\}^h \end{aligned}$$

where $h = \frac{\lambda}{1-\lambda}$, We have for $n > m$,

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n-1}, x_{n-1})G(x_{n-1}, x_{n-2}, x_{n-2}) \cdots G(x_{m+1}, x_m, x_m) \\ &\leq G(x_1, x_0, x_0)^{h^{n-1}} G(x_1, x_0, x_0)^{h^{n-2}} \cdots G(x_1, x_0, x_0)^{h^m} \\ &\leq G(x_1, x_0, x_0)^{h^{n-1}+h^{n-2}+\ldots+h^m} \\ &\leq G(x_1, x_0, x_0)^{\frac{h^m}{1-h}}. \end{aligned}$$

This implies that $G(x_n, x_m, x_m) \to 1$, *as* $n, m \to \infty$, i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$, *when* $m, n \to \infty$. Since

$$\lim_{n,m\to\infty} (G(x_0, x_1, x_1))^{\frac{h^m}{1-h}} = 1,$$

So the sequence$\{x_n\}$ is a Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Since

$$\begin{aligned} G(fz, z, z) &\leq G(fx_n, fz, fz)G(fx_n, z, z) \\ &\leq (G(fz, x_n, x_n)G(fx_n, z, z))^\lambda G(x_{n+1}, z, z) \\ &\leq (G(fz, z, z).G(x_n, z, z)G(x_{n+1}, z, z))^\lambda G(x_{n+1}, z, z), \end{aligned}$$

we have

$$G(fz, z, z) \leq \{G(x_{n+1}, z, z)G(x_n, z, z)\}^\lambda \ G(x_{n+1}, z, z)^{\frac{1}{1-\lambda}} \to 1, (n \to \infty)$$

Hence

$$G(fz, z, z) = 1.$$

This implies that $fz = z$, and thus z is a fixed point of f.

Uniqueness of the fixed point: Suppose that z is another fixed point such that

$$G(u, z, z) = G(fu, fz, fz) \leq (G(fu, z, z)G(fz, u, u))^{\lambda}$$

which implies that $u = z$, and thus the fixed point is unique. The proof is completed. □

Corollary 5. Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f : X \to X$ satisfies the contraction condition

$$G(fx, fy, fy) \leq (G(fx, y, y)G(fy, x, x))^{\lambda}, \ \ for\ \ all\ \ x, y \in \overline{B}(X_o, r),$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda},$$

where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f has a unique fixed point in X, and for any $x \in X$ iterative sequence $(f^n x)$ converges to the fixed point.

Theorem 5. *Let* (X, G) *be a complete multiplicative G-metric space and* $f, g : X \to X$ *a contractive mapping. Then* f *and* g *have a common fixed point if*

$$G(fx, gy, gz) \leq (G(x, y, z))^{\lambda} \ for\ all\ x, y, z \in X\ and\ \lambda \in [0, 1/2).$$

Proof. Let x_0 be any given point in X. Define an iterative sequence $\{x_n\}$ in X such that $x_{2n+1} = f(x_{2n})$ *and* $x_{2n+2} = g(x_{2n+1})$. Now,

$$\begin{aligned}
G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= G(f(x_{2n}), g(x_{2n+1}), g(x_{2n+1}))^{\lambda} \\
&\leq G(x_{2n}, x_{2n+1}, x_{2n+1})^{\lambda} \\
&\leq G(f(x_{2n-1}), g(x_{2n}), g(x_{2n}))^{\lambda} \\
&\leq G(x_{2n-1}, x_{2n}, x_{2n})^{\lambda^2} \\
&\quad . \\
&\leq G(x_0, x_1, x_1)^{\lambda^{n+1}}
\end{aligned}$$

By using triangle inequality, we have

$$\begin{aligned}
G(x_n, x_m, x_m) &\leq (G(x_n, x_{n+1}, x_{n+1}))(G(x_{n+1}, x_{n+2}, x_{n+2})) \cdots (G(x_{m-1}, x_m, x_m)) \\
&\leq (G(x_0, x_1, x_1))^{\lambda^n}(G(x_0, x_1, x_1))^{\lambda^{n+1}} \cdots (G(x_0, x_1, x_1))^{\lambda^{m-1}} \\
&\leq (G(x_0, x_1, x_1))^{\lambda^{n+n+1+\ldots+m-1}} \\
&\leq (G(x_0, x_1, x_1))^{\lambda^n(1+\lambda+\lambda^2+\ldots+\lambda^{m-n-1})} \\
&\leq (G(x_0, x_1, x_1))^{\lambda^n(\frac{1-\lambda^{n-m}}{1-\lambda})}.
\end{aligned}$$

As $\lambda < 1$ and $m, n \to \infty$, so $1-\lambda^{n-m} < 1$, then $G(x_n, x_m, x_m) \leq G(x_0, x_1, x_1)^{\lambda^n}$. This implies that $G(x_n, x_m, x_m) \to 1$ *as* $n, m \to \infty$, i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$. Since,

$$\lim_{n,m\to\infty} (G(x_0, x_1, x_1))^{\lambda^n\left(\frac{1-\lambda^{m-n}}{1-\lambda}\right)} = 1,$$

the sequence$\{x_n\}$ is a Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. So

$$G(x_n, g(u), g(u)) = G(f(x_{n-1}), g(u), g(u)) \leq G(x_{n-1}, u, u)^{\lambda}$$

As $n \to \infty$, so $x_{n-1} \to u$, then

$$G(x_n, g(u), g(u)) = G(u, u, u)^{\lambda}.$$

Thus f and g have a common fixed point u.

Uniqueness of the fixed point: Suppose that z is another fixed point such that

$$G(u, z, z) = G(fu, gz, gz) \leq G(u, z, z)^{\lambda}$$

which implies that $u = z$, and thus the fixed point is unique. The proof is completed. □

Corollary 6. Let (X, G) be a complete multiplicative G-metric space and $f, g : X \to X$ be a contractive mapping. Then f and g have a common fixed point if

$$G(fx, gy, gz) \leq (G(x, y, z))^{\lambda} \text{ for all } x, y, z \in \bar{B}(x_0, r),$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda},$$

where $\lambda \in [0, 1/2)$.

Theorem 6. *Let (X, G) be a complete multiplicative G-metric space and $f, g : X \to X$ be a contractive mapping. Then f and g have a common fixed point if*

$$G(fx, gy, gy) \leq (G(x, y, y))^{\lambda}, \text{ for all } x, y \in X \text{ and } \lambda \in [0, 1/2).$$

Proof. Let x_0 be any given point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = f(x_{2n}) \text{ and } x_{2n+2} = g(x_{2n+1}).$$

Now

$$\begin{aligned} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= G(f(x_{2n}), g(x_{2n+1}), g(x_{2n+1})) \\ &\leq G(x_{2n}, x_{2n+1}, x_{2n+1})^{\lambda} \\ &\leq G(f(x_{2n-1}), g(x_{2n}), g(x_{2n}))^{\lambda} \end{aligned}$$

$$\leq G(x_{2n-1}, x_{2n}, x_{2n})^{\lambda^2}$$

.

$$\leq G(x_0, x_1, x_1)^{\lambda^{n+1}}$$

By using triangle inequality, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq (G(x_n, x_{n+1}, x_{n+1})).(G(x_{n+1}, x_{n+2}, x_{n+2}))\ldots(G(x_{m-1}, x_m, x_m)) \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n}.(G(x_0, x_1, x_1))^{\lambda^{n+1}}\ldots(G(x_0, x_1, x_1))^{\lambda^{m-1}} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n+\lambda^{n+1}+\lambda^{n+2}+\ldots+\lambda^{m-1}} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n(1+\lambda+\lambda^2+\ldots+\lambda^{m-n-1})} \\ &\leq (G(x_0, x_1, x_1))^{\lambda^n(\frac{1-\lambda^{n-m}}{1-\lambda})}. \end{aligned}$$

As $\lambda < 1$ and $m, n \to \infty$, so $1 - \lambda^{n-m} < 1$, then

$$G(x_n, x_m, x_m) \leq G(x_0, x_1, x_1)^{\lambda^n}.$$

This implies that $G(x_n, x_m, x_m) \to 1$ *as* $n, m \to \infty$, i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$. Since

$$\lim_{n,m\to\infty} (G(x_0, x_1, x_1))^{\lambda^n(\frac{1-\lambda^{n-m}}{1-\lambda})} = 1,$$

the sequence $\{x_n\}$ is a Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $u \in X$ such that $\lim_{n\to\infty} x_n = u$. So

$$G(x_n, g(u), g(u)) = G(f(x_{n-1}), g(u), g(u)) \leq G(x_{n-1}, u, u)^{\lambda}.$$

As $n \to \infty$, so $x_{n-1} \to u$, then

$$G(x_n, g(u), g(u)) = G(u, u, u)^{\lambda}.$$

Thus f and g have a common fixed point u.

Uniqueness of the fixed point: Suppose that z is another fixed point such that

$$G(u, z, z) = G(fu, gz, gz) \leq G(u, z, z)^{\lambda},$$

which implies that $u = z$, and thus the fixed point is unique. The proof is completed. □

Corollary 7. Let (X, G) be a complete multiplicative G-metric space and $f, g : X \to X$ be a contractive mapping. Then f and g have a common fixed point if

$$G(fx, gy, gy) \leq (G(x, y, y))^{\lambda}, \ \textit{for all } x, y \in \overline{B}(X_0, r),$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda},$$

where $\lambda \in [0, 1/2)$.

Theorem 7. *Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f,\ g : X \to X$ satisfies the contraction condition*

$$G(fx, gy, gy) \le (G(fx, x, x)G(gy, y, y))^{\lambda}$$

for all $x, y \in X$ where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f and g has a common fixed point in X.

Proof. Let x_0 be any given point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = f(x_{2n}) \ and \ x_{2n+2} = g(x_{2n+1}).$$

Thus we have

$$\begin{aligned} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= G(fx_{2n}, gx_{2n+1}, gx_{2n+1}) \\ &\le (G(fx_{2n}, x_{2n}, x_{2n})G(gx_{2n+1}, x_{2n+1}, x_{2n+1}))^{\lambda} \\ &= (G(x_{2n+1}, x_{2n}, x_{2n})G(x_{2n+2}, x_{2n+1}, x_{2n+1}))^{\lambda}, \end{aligned}$$

and

$$\begin{aligned} G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &\le (G(x_{2n}, x_{2n+1}, x_{2n+1}))^{\frac{\lambda}{1-\lambda}} \\ &= \{G(x_{2n}, x_{2n+1}, x_{2n+1})\}^{h}, \end{aligned}$$

where $h = \frac{\lambda}{1-\lambda}$ By using Triangle inequality, we have for $n > m$,

$$\begin{aligned} G(x_n, x_m, x_m) &\le G(x_n, x_{n-1}, x_{n-1})G(x_{n-1}, x_{n-2}, x_{n-2}) \cdots G(x_{m-1}, x_m, x_m) \\ &\le G(x_1, x_0, x_0)^{h^{n-1}} G(x_1, x_0, x_0)^{h^{n-2}} \cdots G(x_1, x_0, x_0)^{h^{m}} \\ &\le G(x_1, x_0, x_0)^{h^{n-1}+h^{n-2}+\ldots+h^{m}} \\ &\le G(x_1, x_0, x_0)^{\frac{h^m}{1-h}}. \end{aligned}$$

This implies that

$$G(x_n, x_m, x_m) \to 1 \ as \ n, m \to \infty,$$

i.e., $\lim_{m,n\to\infty} G(x_n, x_m, x_m) = 1$. Since

$$\lim_{n,m\to\infty} (G(x_0, x_1, x_1))^{\frac{h^m}{1-h}} = 1,$$

the sequence$\{x_n\}$ is a Cauchy sequence.

Existence of the fixed point: By the completeness of X, there is a $z \in X$ such that $x_n \to z$ $(n \to \infty)$. Since

$$\begin{aligned} G(fx, gz, gz) &\le G(fx_n, gz, gz)G(fx_n, z, z) \\ &\le (G(fx_n, x_n, x_n)G(gz, z, z))^{\lambda} G(x_{n+1}, z, z) \end{aligned}$$

we have

$$G(fz, gz, gz) \le (G(fx_n, x_n, x_n)^{\lambda} \ G(x_{n+1}, z, z))^{\frac{1}{1-\lambda}} \to 1 \ (n \to \infty).$$

Hence $G(fz, gz, gz) = 1$. Thus f and g have a common fixed point z.

Uniqueness of the fixed point: Suppose that z is another fixed point such that

$$G(u, z, z) = G(fu, gz, gz) \leq \{f(fu, u, u).G(gz, z, z)\}^{\lambda},$$

which implies that $u = z$, and thus the fixed point is unique. The proof is completed. □

Corollary 8. Let (X, G) be a multiplicative G-metric space. Suppose that the mapping $f, g : X \to X$ satisfies the contraction condition

$$G(fx, gy, gy) \leq (G(fx, x, x)G(gy, y, y))^{\lambda}, \ \text{for all } x, y \in \overline{B}(x_0, r)$$

and

$$G(x_0, f(x_0), f(x_0)) \leq r^{1-\lambda},$$

where $\lambda \in [0, \frac{1}{2})$ is a constant. Then f and g has a common fixed point in X.

4. Application of Multiplicative G-Metric Space in Travelling Salesman Problem

Travelling Salesman Problem: This is very well known problem of graph theory. In this problem, a person who is selling different goods to different cities. He wants to go to all cities at the same day but in a less time by using shortest path. Here, we are giving C++ code as an application of multiplicative G-metric space by using this code we calculated shortest path for salesman who can visit all cities. We are giving this code for 10 cities. To solve the Traveling Salesman Problem (TSP) using the multiplicative G-metric space in C++, we need to consider a slight modification in the distance calculation compared to the regular Euclidean or geometric distance. In a multiplicative G-metric space, the distance between two cities is calculated as the product of individual distances along the path, each raised to a power G. This code calculates the distance between cities based on the multiplicative G-metric space and finds the shortest path using a similar approach.

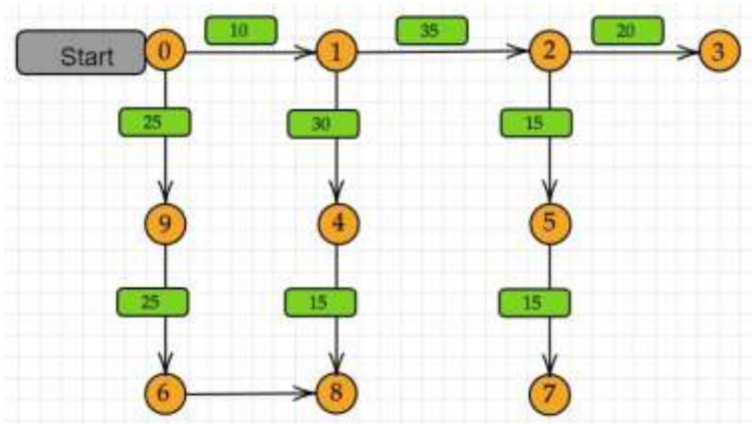

Figure 1: Travelling Salesman Problem of 10 Cities

```
#include <iostream>
#include <vector>
#include <algorithm>
#include <limits>
#include <cmath>

using namespace std;
// Class: Multiplicative G-Metric Space
// Define the number of cities
const int number_cities = 10;

// Define the distance matrix (adjacency matrix)
int distances[number-cities][number-cities] = {
    {0, 10, INT-MAX, INT-MAX, 25, INT-MAX, INT-MAX, INT-MAX,
    INT-MAX, INT-MAX},
    {10, 0, 35, 30, INT-MAX, INT-MAX, INT-MAX, INT-MAX,
    INT-MAX, INT-MAX},
    {INT-MAX, 35, 0, 20, INT-MAX, INT-MAX, INT-MAX, INT-MAX,
    INT-MAX, INT-MAX},
    {INT-MAX, 30, 20, 0, INT-MAX, INT-MAX, INT-MAX, INT-MAX,
    INT-MAX, INT-MAX},
    {25, INT-MAX, INT-MAX, INT-MAX, 0, 15, INT-MAX, INT-MAX,
    INT-MAX, INT-MAX},
    {INT-MAX, INT-MAX, INT-MAX, INT-MAX, 15, 0, INT-MAX,
    INT-MAX, INT-MAX, INT-MAX},
    {INT-MAX, INT-MAX, INT-MAX, INT-MAX, INT-MAX, INT-MAX,
    0, INT-MAX, INT-MAX, 15},
    {INT-MAX, INT-MAX, INT-MAX, INT-MAX, INT-MAX, INT-MAX,
    INT-MAX, 0, INT-MAX, INT-MAX},
```

```
    {INT_MAX, INT_MAX, INT_MAX, INT_MAX, INT_MAX, INT_MAX,
     INT_MAX, INT_MAX, 0, INT_MAX},
    {INT_MAX, INT_MAX, INT_MAX, INT_MAX, INT_MAX, INT_MAX,
    15, INT_MAX, INT_MAX, 0}
};

// Define the power for the G−metric
const double G = 1.5; // You can adjust this value as needed

// Function to calculate the distance between two cities
in the multiplicative G−metric space
double distance__between(int city1, int city2) {
    // If either city is unreachable, return INT_MAX
    if (distances[city1][city2] == INT_MAX ||
    distances[city2][city1] == INT_MAX)
        return INT_MAX;

    // Calculate distance using the multiplicative G−metric formula
    return pow(distances[city1][city2] * distances[city2][city1], G);
}

// Function to calculate the total distance of a path
double total__distance(const vector<int>& path) {
    double total = 1; // Initialize total distance

    for (size_t i = 0; i < path.size() − 1; ++i) {
        total *= distance__between(path[i], path[i + 1]);
    }
    // Return to the starting city
    total *= distance__between(path.back(), path.front());

    return total;
}

int main() {
    // Vector to store the order of cities visited
    vector<int> path_(number_cities);
    for (int i = 0; i < number_cities; ++i) {
        path_[i] = i;
    }

    // Find the shortest path
    double mini__distance = numeric_limits<double>::max();
    vector<int> short_path;
    do {
        double current_distance = total__distance(path_);
```

```
        if (current_distance < mini__distance) {
            mini_distance = current_distance;
            short_path = path_;
        }
    } while (next_permutation(path_.begin() + 1, path_.end())); //
    Start from the second city

    // Output the shortest path
    cout << "Shortest path: ";
    for (int city : short_path) {
        cout << city << " ";
    }
    cout << endl;
    cout << "Shortest distance: " << mini__distance << endl;

    return 0;
}
```

5. Conclusion

We proved some fixed point theorems by taking settings of multiplicative *G*-metric spaces. We also proved theorems by settings of two maps, and also gave corollaries of Closed balls in the settings of multiplicative *G*-metric spaces. Still, multiplicative *G*-metric space has to be explored further with its applications in Science and technology. At the end we have given an application of Multiplicative *G*-metric space in travelling salesman problem to find shortest path and shortest distance for salesman. By using this code we can find shortest route between any number of cities. Our application of multiplicative *G*-metric space is more robust than existing techniques because other methods are time consuming but our technique is faster than other existing methods. It looks that the further research in this field will open a new gate for researchers.

Conflict of interests

The authors declare that they have no competing interests regarding the publication of this paper.

Data availability

The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' contributions:

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements

This work was partly supported by Yunnan Fundamental Research Projects (No. 202401AT070479).

References

References

[1] M. Fr´echet. M, Sur quelques points du calcul fonctionnel. Rendiconti del Circolo Matematico di Palermo. 22 (1906) 1-72.

[2] S. Banach, Sur les op´erations dans les ensembles abstraits et leur applications aux ´equations int´egrales, Fund. Math. 3 (1922) 133-181.

[3] A. Alrazi, J. Ahmad, L-Fuzzy Mappings and Common Fixed Point theorems, Nonlinear functional Analysis and Applications. 23 (2018) 661-672.

[4] A. Azam, I. Beg, Common fixed points of fuzzy maps, Mathematical and Computer Modeling. 49 (2009) 1331-1336.

[5] A. Azam, Fuzzy Fixed points of Fuzzy mappings via a Rational inequality, Hacetepe Journal of Mathematics and Statistics. 40 (2011) 421-431.

[6] P. Nagpal, S. Kumar, S.K. Garg, Fixed point results in multiplicative generalized metric spaces, Advances in Fixed Point Theory. 6 (2016) 352-386.

[7] M. Mazhar, M. Bilal, S. Abdullah, Rational Type Contraction Mapping Theorems on Multiplicative *G*-Metric Spaces,Journal of science and arts. 22 (2022) 605-618.

[8] S. Gahler, 2-nlet,riclie raume urid ihre topologische strukture, Math. Nachr. 26 (1963) 115-148.

[9] S. Gahler, Zur geometric 2-metriche raume, Reevue Roumaine de Math. Pures at Appl. XI (1966) 664-669.

[10] B.C. Dhage, Generalized metric spaces and mapping with fixed points, Bull. Cal. Math. Soc. 84 (1992) 329-336.

[11] B.C. Dhage, A common fixed point principle in *D*-metric space, Bull. Cal. Math. Soc. 91 (1999) 475-480.

[12] B.C. Dhage, Some results on common fixed points I, Indian J. Pure Appl. Math. 30 (1999) 827-837.

[13] B.C. Dhage, Generalized Metric Space and Topological Structure I, An.stiint, Univ. Al I.Cuza Iasi. Mat. 46 (2000) 3-24.

[14] Z. Mustafa, B. Sims, Some Remarks Concerning *D*-Metric Spaces, proceedings of the International Conferences on F.Pt.Th. and App, Valencia (spain), 2003: 189-198.

[15] Z. Mustafa, B. Sims, A new approach to a generalized metric space. J. Nonlinear Convex Anal. 7 (2006) 289-297.

[16] Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in com- plete *G*-metric spaces, Fixed point theory and Applications, 2009 (2009) 917175.

[17] Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorems for mappings on complete *G*-metric spaces, Fixed point theory and Applications, 2008 (2008) 189870.

[18] Z. Mustafa, W. Shatawani, M. Bataineh, Fixed point theorems on complete G-metric spaces, Journal of Mathematics and Statistics, 4 (2008) 196-201.

[19] M. Ö zavasar, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, Journal of Engineering Technology and Applied Sciences, 2 (2017) 65-79.

[20] A. Bashirov, E. Kurpinar, A. Ö zyapici, Multiplicative Calculus and its applications. J. Math. Anal, Appl. 337 (2008) 36-48.

[21] M. Abbas, B. Ali, Y. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with applications. International Journal of Mathematics and Mathematical Sciences, 2015 (2015) 218683.

[22] B. C. Dhage, A study of some fixed point theorems, Ph.D. thesis, Marathwada University, Aurangabad, India, 1984.