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Solving A Free-Underdamped Differential Equation and Using Its Computer-Generated Graph to Explain the Acoustical Response of a Loudspeaker

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ABSTRACT:

The effective power of an image as a teaching tool is widely known. Furthermore, when a graph is also used to describe a physical phenomenon along with its mathematical underpinning the explanation always seems better and clearer. Searching for mathematical formulas which graphs match or closely resemble a particular physical behavior is a desirable pedagogical tool to convey the essence of a technical work and make it more understandable. In this respect, mathematical formulas and their graphs serve not only as a channel to teach complex phenomena but as a shortcut to avoid talking separately about diverse cases of similar nature. In this paper the authors consider the solution of a free-underdamped differential equation and its computer-generated graph to study the directivity of a loudspeaker.

I. INTRODUCTION

Using the graph of a particular mathematical function to resemble an acoustical event has always been a valuable aid to researchers when studying and measuring acoustical phenomena. When testing the differences of microphones and loudspeakers is very common to make use of polar coordinates. For example, to study a microphone's pattern of sensitivity engineers test its output signal by measuring it at different angles (see Figure 1). That is, an acoustical signal is sent to the microphone while varying the source angle and measuring the corresponding output level. Microphone sensitivity is used to determine the pick-up level of the sound from different directions. A curve frequently used to describe a directional microphone – it could be omnidirectional, bidirectional, or unidirectional - is that of a cardioid. Figure 2 shows the graph of a typical cardioid generated using Maple™.

Figure 1 Method to determine the sensibility pattern of a microphone.

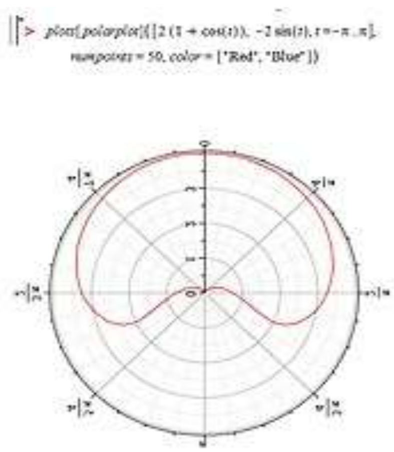

Figure 2 Computer-generated graph of a cardioid.

Comparing the computer-generated graph of Figure 2 with the ones provided by a manufacturer - determining the sensitivity of a Shure™ SM-57 microphone - we can observe their similitude (See Figure 3) [1].

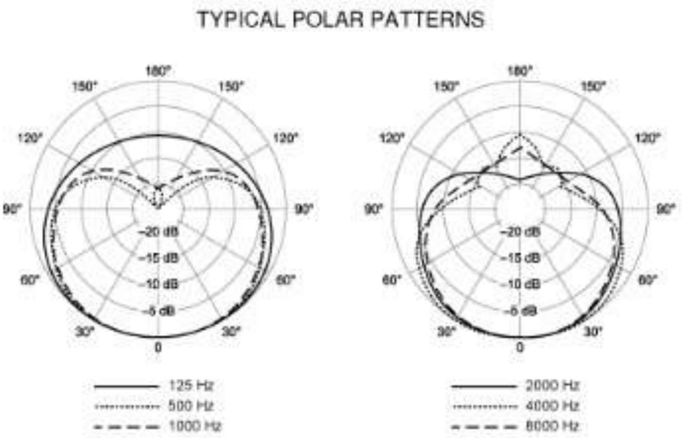

Figure 3 Polar pattern of a Shure™ SM-57 Microphone

Another example of a mathematical formula that can be used to simulate an acoustic event such as the reverberation time in a closed-model room is given by the function $f(x) = x\sin(1/x)$ (See Figure 4) [2].

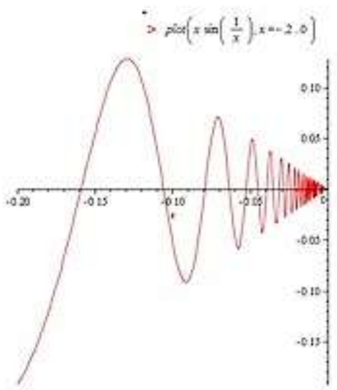

Figure 4 Curve that simulates a reverberation time of a closed-model room.

Comparing the preceding graph with the one of a real test of reverberation time (See figure 5) we can observe again their similarities [3].

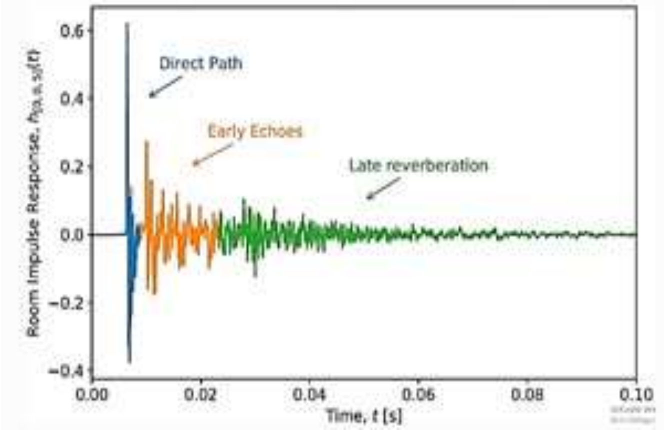

Figure 5 Measure of reverberation time in a closed-model room.

For the remaining of this article, we will consider first the general solution of a second-order homogeneous differential equation using an algebraic method [4]. Second, we will use this method on a sample differential equation and, from this solution, generate a solid of revolution that will be used to illustrate its relation to the response of a loudspeaker. These two activities will allow us to exemplify how a formula, its graph, and its mathematical underpinning can be used to clarify the understanding of a physical event.

1. GENERAL RESOLUTION OF A SECOND-ORDER HOMOGENEOUS DIFFERENTIAL EQUATION BY AN ALGEBRAIC METHOD

The general form of a second-order homogenous differential equation where a, b, and c are arbitrary constants is as follows

$ay" + by' + cy = 0$ (1)

For this type of equation, the function $y = e^{mx}$ is a solution as we will show next. The first and second derivatives of this function are

$y' = me^{mx}$ $y" = m^2e^{mx}$

Substituting these last two expressions and the value of the function back into (1) we obtain

$$a\,(m^2e^{mx}) + b(e^{mx}) + c(e^{mx}) = 0$$

Factoring common terms, we can write the equality as

$e^{mx}(am^2 + bm + c) = 0$ (1-A)

Because $e^{mx} \neq 0$ for any real x, equation (1-A) is satisfied when m is a root of its auxiliary quadratic equation. That is, when

$am^2 + bm + c = 0$ (2)

Without loss of generality, we can assume that the roots $m1$, $m2$ of (2) are conjugate complex numbers. Thus, let's assume that

$m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

where α, β are real numbers and $\beta > 0$.

Replacing m_1 and m_2 in $y = e^{mx}$ we can say that a

general solution for (1) is of the form

$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$ (3)

Distributing the exponents and applying the exponential properties we can write

$y = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$ (3-A)

To avoid working with complex exponentials we can use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ (θ is real) and the fact that cos(-βx) = cos(βx) and sin(-βx) = - sin(βx). Therefore, we can rewrite $e^{i\beta x}$ and $e^{-i\beta x}$ as shown next.

$e^{i\beta x} = \cos\beta x + i\sin\beta x$ and $e^{-i\beta x} = co\,s\,\beta x - i\,sin\,\beta x$

replacing these values in (3-A) and factoring common terms we can obtain

$$y = e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C2(\cos\beta x - i\sin\beta x)]$$

Applying the distributive law to this last equality we have that

$$y = e^{\alpha x}(C_1\cos\beta x + C_1 i\sin\beta x + C_2\cos\beta x - C_2 i\sin\beta x)$$

Factoring the common terms results in

$$y = e^{\alpha x}[(C_1 + C_2)\cos\beta x + (C_1 i - C_2 i)\sin\beta x]$$

From this result, we can say that $e^{\alpha x}$cos(βx) and $e^{\alpha x}$sin(βx) are real solutions of (1) over the range (-∞,+∞). Renaming ($C_1 + C_2$) and (C_1i - C_2i) as c_1 and c_2 respectively, the general solution of (1) can be rewritten as

$$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

1.1 GENERAL SOLUTION OF A SECOND-ORDER HOMOGENEOUS DIFFERENTIAL EQUATION OF A FREE UNDERDAMPED SYSTEM

The general form of a second-order homogeneous differential equation of a is [5]

$\frac{d^2x}{td^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$ (5)

Following the procedure of Section 1.1 and applying it to equation (5) where $a = 1$, $b = 2\lambda$, and $c = \omega^2$, we can write directly its associated auxiliary quadratic equation as

$m^2 + 2\lambda m + \omega^2 = 0$ (6)

using the general formula for solving (6) and performing the basic substitutions we obtain

$$\frac{-2\lambda \pm \sqrt{(2\lambda)^2 - 4(\omega)^2}}{2}$$

simplifying this last expression, the roots are:

$$m1 = -\lambda + \sqrt{\lambda^2 - \omega^2} \qquad m2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

or more succinctly

$$-\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

Also as indicated in [5], for a free-underdamped system it is necessary that the discriminant be of the following form

$$\lambda^2 - \omega^2 < 0$$

Therefore, the roots of the auxiliary equation will be conjugated complex and can be written as

$$m1 = -\lambda + \sqrt{\lambda^2 - \omega^2}\, \boldsymbol{i} \qquad m2 = -\lambda - \sqrt{\lambda^2 - \omega^2}\, \boldsymbol{i}$$

In consequence, the general solution for (5) is

$$x(t) = e^{-\lambda t}[c_1 \cos\sqrt{w^2 - \lambda^2}\, t + c_2 \sin\sqrt{w^2 - \lambda^2} t]$$

1.2 A LOUDSPEAKER AS AN UNDERDAMPED SYSTEM

A loudspeaker can be considered as a mass-spring-damper system. Figure 6 shows a simplified view of a loudspeaker [6]. Jointly, the cone (the body) with the voice coil represents the moving mass. The suspension, formed by the spider and the ring surround, keep the moving parts in place (with a degree of freedom) and provides the rigidity, mechanical damping, and resistance of the system. The general equation for a system of this type is given by the following second-order differential equation

$$F = M\frac{d^2x}{dt} + D\frac{dx}{dt} + Kx = 0 \qquad (7)$$

where *F* is the force (pressure) applied to the voice coil, *x* is the cone displacement, *K* the suspension stiffness, *M* is the moving mass, and *D* the mechanical damping or resistance.

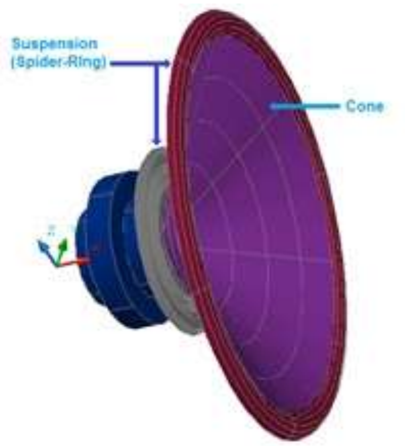

Figure 6 Low frequency loudspeaker parts.

1.4 OUR SAMPLE DIFFERENTIAL EQUATION

In the pedagogical spirit of this paper, let's consider the following particular case of the aforementioned equation (7)

$\frac{d^2y}{dx} + 2\frac{dy}{dx} + 26y = 0$ (8)

Applying the algebraic method for the resolution of its associated auxiliary quadratic equation to (8) we have

$y = \frac{-2\pm\sqrt{2^2-4(1)(26)}}{2(1)}$ (9)

Performing operations results in

$$y = 1 \pm \frac{\sqrt{100}\,\sqrt{-1}}{2}$$

Simplifying this last expression, we obtain

$$y = -1 \pm 10i$$

1.4.1 SOLVING OUR SAMPLE DIFFERENTIAL EQUATION

We now demonstrate that

$y = e^{-x}(\cos 5x + \sin 5x)$

is a solution of (8). Following a similar procedure to solve equation (1) we proceed as indicated next.

First we obtain the first derivative, y' and double it. The result of these operations is

$$2y' = -2e^{-x}(\cos 5x + \sin 5\, x) + 2e^{-x}(-5 \sin 5\, x + 5 \cos 5x)$$

$$2y' = -2e^{-x} \cos 5x - 2e^{-x} \sin 5x - 10e^{-x} \sin 5x + 10e^{-x} \cos 5x$$

$2y' = 8\, e^{-x} \cos 5x - 12e^{-x} \sin 5$ (9)

The second derivative, $y"$, is shown next

$$y" = e^{-x}(\cos 5x + \sin 5\, x) - e^{-x}(-5 \sin 5x + 5 \cos 5\, x) - e^{-x}(-5 \sin 5x + 5 \cos 5x) + e^{-x}(-25 \cos 5\, x - 25 \sin 5\, x)$$

$$y^{"} = e^{-x} \cos 5x + e^{-x} \sin 5\, x + 5e^{-x} \sin 5x - 5e^{-x} \cos 5x + 5e^{-x} \sin 5 - 5e^{-} \cos 5 - 25e^{-x} \cos 5x - 25e^{-x} \sin 5x$$

$y^{"} = -34e^{-x} \cos 5x - 14e^{-x} \sin 5x$ (10)

Replacing (9) and (10) back into (8) results in

$$y^{"} + 2y' + 26y = -34e^{-x} \cos 5x - 14e^{-x} \sin 5x + 8e^{-x} \cos 5x - 12e^{-x} \sin 5x + 26e^{-x} \cos 5 + 26e^{-x} \sin 5x = 0$$

As an aid to the reader, the following table shows the coefficients of the sine and cosine terms of the previous expression and their algebraic sum (Total column).

				Total
cos(5x)	-34	8	26	0
sin(5x)	-14	-12	26	0

II. GRAPHING OUR SOLUTION WITH MAPLE

One of the main goals of this work is on obtaining mathematical formulas and their graphs that simulate a known physical event. To illustrate this purpose using a variant of the previous example we have manipulated the exponential variable (parameter) of (11) to obtain high and low-mid frequency responses.

2.1 EMULATING A HIGH FREQUENCY RESPONSE

When plotted, as shown in Figure 7, the curve generated by the function

$y = e^{-2x}(\cos 5x + \sin 5x)$ (11)

resembles a high frequency response.

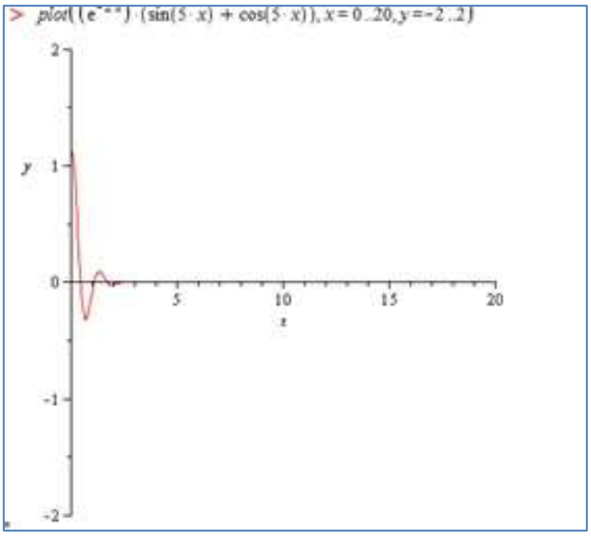

Figure 7 Plot for function *y = e*$^{-2x}$(sin(5*x*) + co*s*(5*x*))

The solid of revolution for equation (11) obtained by using the Maple Calculus-1 tutorial of Figure 8 is

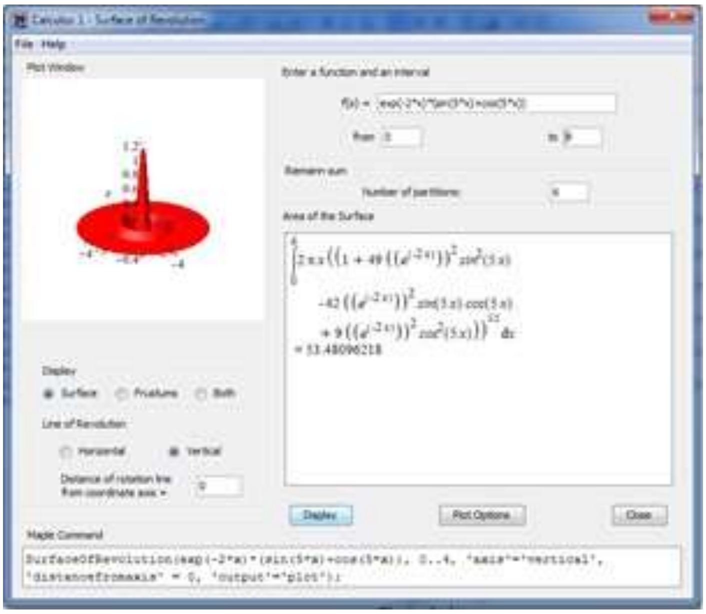

Figure 8 Solid of revolution for function

***y* = *e*$^{-2x}$(sin(5*x*) + cos(5*x*))**

Let's consider now the computer-generated graph of Figure 8 with a real one provided by a manufacturer (See Figure 9). Comparing the shape of the curve of the red solid of revolution (See Figure 8) with the light-blue shape at the center of Figure 9 we can see that the light-blue curve corresponding to a frequency of 5000 Hz resembles the vertical component of Figure 8.

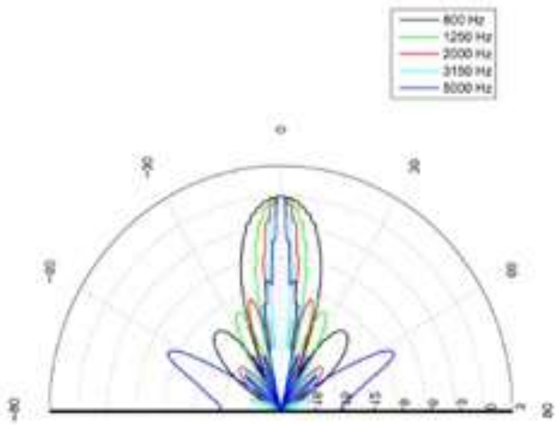

Figure 9 Loudspeaker directivity polar pattern at different frequencies [7].

Figure 10 shows a loudspeaker spectrum in 3d. from this graph we can observe the variation of the spectrum if the frequency is lowered from 5000Hz to 3500Hz.

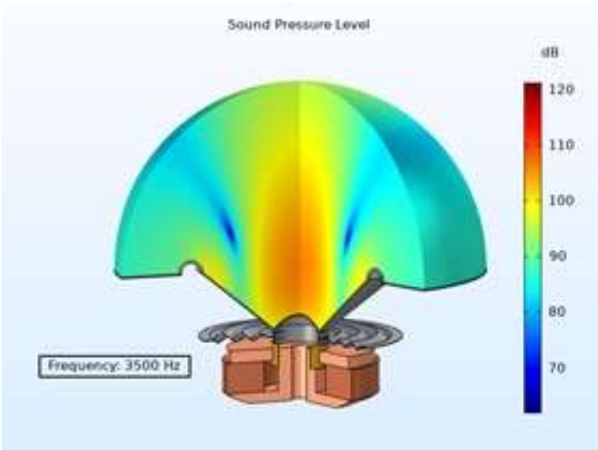

Figure 10 A 3D Loudspeaker frequency spectrum at 3500Hz [8].

2.2 EMULATING A MID-LOW FREQUENCY RESPONSE

For our next example let's use the function shown below to mimic a frequency response. Figure 11 shows the plotting of this function.

$y = e^{-0.9x}(\text{Sin}(5x) + \text{Cos}(5x))$

Figure 12 shows the solid of revolution generated by this function using the Maple.

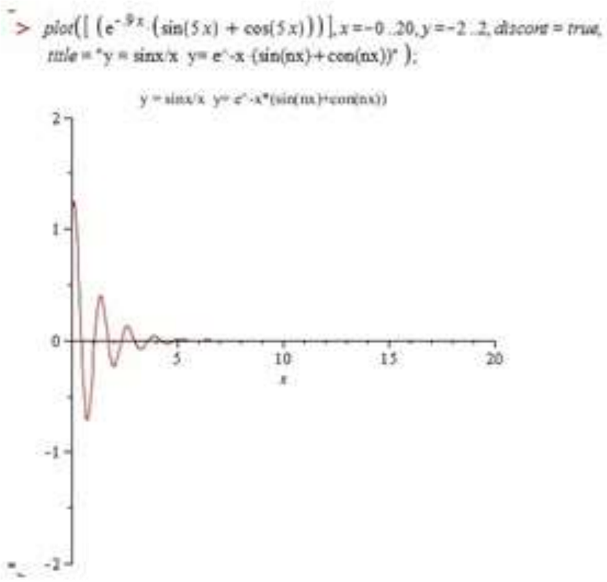

Figure 11 Graph of function $y = e^{-0.9x}$ (sin(5x) + cos(5x)).

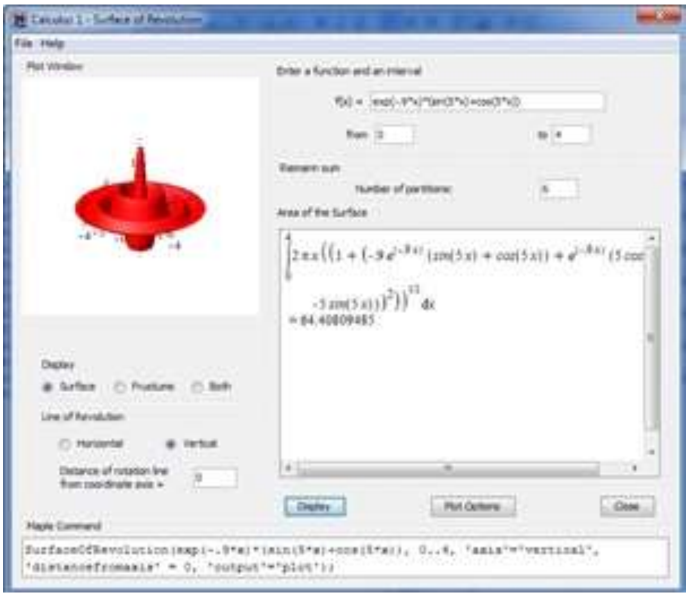

Figure 12 solid of revolution generated for function

***y* = *e*$^{-0.9x}$ (Sin(5x) + Cos(5x)).**

If we now compare the graphic of Figure 12 with the one of a regular Mid-Low frequency loudspeaker (See Figure 13) we do not find similarities like we did in section 2.1. The reason for this is that lowering the frequency makes the sound less directional. Notice that the main vertical lobes in Figure 13 do not have narrow shapes. But if we take a graphic of loudspeakers of latest technologies or its arrays, like correlated acoustical sources, Line Array Systems or Cardioid Arrays on subwoofers (See Figure 14) we find again the similarities with the narrow shapes in the center of the spectrum.

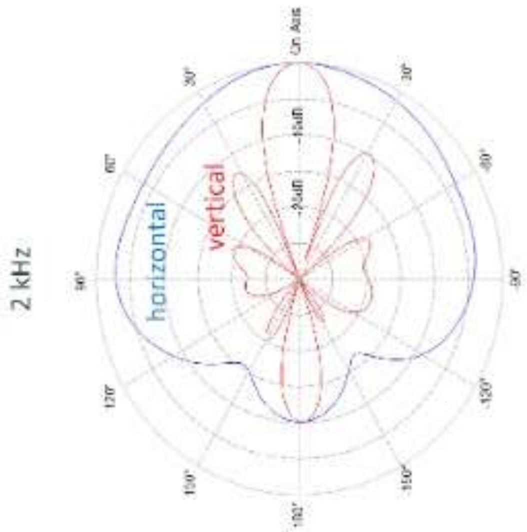

Figure 13 Vertical and horizontal polar pattern of a JBL™ loudspeaker at 2 KHz [9].

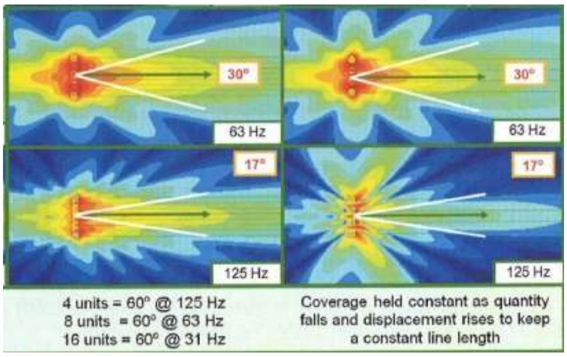

Figure 14 Spacing effects – "steering" a coupled line source arrays of loudspeakers [10].

III. OTHER APPLICATIONS

The use of second-order homogeneous differential equations of the type considered in this work to explain graphically physical events is not limited to acoustics phenomena only. As shown in Figure 15 the same ideas can be applied to studying the mechanics of a trampoline [11]. Additional areas of applications to consider may include fluid mechanics such as those of hose or spray guns (See Figure 16).

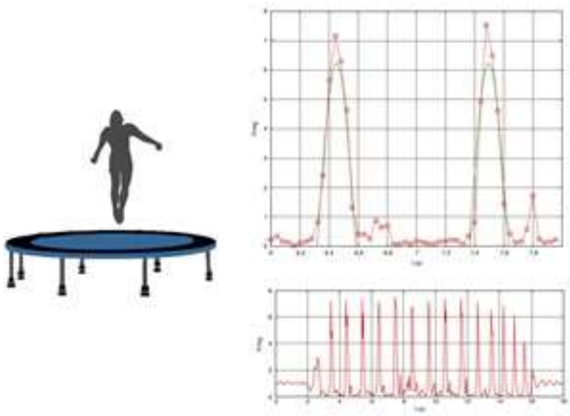

Figure 15 Comparison between theoretical results and accelerometer data for bounces on a 1.3 mt. circular trampoline (Sampling Rate 25Hz)

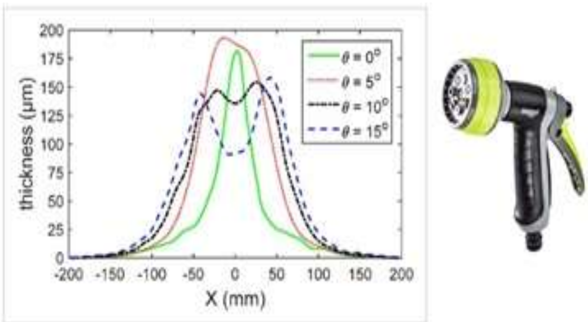

Figure 16 Coating thickness profiles along the x-axis for different θ values when L = 30 mm [11].

IV. CONCLUSION

The results obtained in this work show that use of differential equations to better explain physical events are helpful. Firstly, differential equations can be used then as a pedagogical tool with a four-fold purpose. First, as a methodology to exemplify the use of mathematics to find or illustrate similarities between simulated and real physical phenomena. Secondly, it may serve as a stimulating factor for students who, through the use of graphs and differential equations, can make their research experience more enjoyable. Thirdly, by obtaining ratios between the equation values to radius, heights, radiations, angles, frequencies, and so on, this approach can be useful too. Fourthly, it is important to mention that other areas of applications, where computer graphs and animations may provide valuable insights, may include - but not limited to - fluid mechanics, illumination, air flow, gas expansion etc. Finally, we want to encourage the use of differential equations for different goals and uses for which they were originally created because it may open doors to new uses or discoveries.

V. REFERENCES

[1] Shure, SM57 Instrument Microphone User Guide, Shure Inc., 2023 https://pubs.shure.com/guide/SM57/en-US

[2] Maple Software plot. https://www.maplesoft.com/

[3] Di Carlo, D.., Tandeitnik, P., Foy, C., Bertin, N.; Deleforge, A.; Gannot, S. dEchorate: a calibrated room impulse response dataset for echo-aware signal processing, Eurasip Journal On Audio, 2021. https://asmp-eurasipjournals.springeropen.com/articles/10.1186/s13636-021-00229-0

[4] Zill, D. A First course in Differential Equations with Applications, 2nd. Edition, PWS Publishers, Wadsworth, Inc. Belmont, California, 1982.

[5] Zill, D. Advance Engineering Mathematics, 7th Edition, Jones & Barlett Learning, Burlington, MA, 2022.

[6] Paavo Jumppanen, L. Parameter Of A Dynamic Loudspeaker, Har-Bal, 2013.

[7] JBL Professional CBT Constant Beamwidth Technology, Technical Notes Volume 1, Number 35, 2009.

[8] Christopher, B. Samsung Amps Up Loudspeaker Designs with Simulation, COMSOL, 2019, https://www.comsol.com/blogs/samsung-amps-up-loudspeaker-designs-with-simulation/

[9] Polar Far-Field Measurement (POL), 2023 https://www.klippel.de/products/rd-system/modules/pol-polar-far-field-measurement.html.

[10] Meyer, D. G. Sound Reinforcement System Design, ECE 40020, 2020.

[11] Pendrill, A-M., Eager, D. (2015). Free fall and harmonic oscillations - analyzing trampoline jumps. Physics Education, (50), 64-70. https://doi.org/10.1088/0031-9120/50/1/64.

[12] Wang, Y-A.; X-P, Xie.; Lu, X-H. Design of a Double-Nozzle Air Spray Gun and Numerical Research in the Interference Spray Flow Field. Coatings 2020, 10, 475. https://doi.org/10.3390/coatindgs10050475

TRADEMARKS

Maple is mathematical software and trademark of Maplesoft.

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