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Valuation of an Asset Using a Skewed Random Pricing Tree with Fractional Dynamics

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Abstract: *Inspired by previous models in the literature, this project focuses on creating a tree pricing model utilizing underlying asset price dynamics governed by Itô-Mckean skew fractional Brownian motion (FBM). The objective is to simplify the Black-Scholes option pricing equation, which is modelled by fractional Brownian motion, into a one-dimensional heat equation. This simplification enables obtaining the solution through the application of the Laplace transform method (LTM). Additionally, the project includes conducting a sensitivity analysis for the worth profile, accompanied by illustrative examples within a specific context. The researcher possesses knowledge of the binomial method of option pricing as well as tree theory.*

Keywords: *Worth Profile; FBM; Binomial Tree; Option pricing.*

I. INTRODUCTION

In the realm of dynamic asset pricing theory, discrete-time option pricing models have traditionally been overshadowed by their continuous-time counterparts ([Bock & Korn, 2016; Shirvani et al., 2020) [1, 2]. This is primarily attributed to the well-developed theory surrounding semi-martingales, Lévy processes, and the fundamental theorem of asset pricing (Jarrow, 2012; Protter, 2004; Guasoni et al., 2012) [3, 4, 5]. Options are among the most widely utilized financial products, enabling the trading of future market prices, bond futures, currency futures, commodity equities, and interest rate futures (Li, 2019)[6] . The Black–Scholes Model is a popular choice for option pricing, representing one of the most critical applications in finance. In the absence of transaction costs, the value of an option can be determined using the Black–Scholes model. In a related context, Caputo, Vijayan, & Manimaran (2023) [7] proposed a solution for the fractional Black–Scholes equation (FBSE) problem.

Their primary objective was to demonstrate the solution to the fractional Black–Scholes equation (FBSE) using a semi-analytical method known as the homotopy analysis Shehu transform method (HASTM). They also conducted a comparative analysis with other methods such as the homotopy analysis method (HAM), homotopy perturbation method (HPM), and Elzaki transform homotopy perturbation method (ETHPM). Fractional calculus has seen increased use in analyzing stochastic processes driven by fractional Brownian motion processes (Osu & Chukwunezu, 2016)[8]. Fractional Brownian motion with a Hurst exponent H ∈ (0,1) is a stochastic process $\{B_R(t), t \in \mathbb{R}\}$ that satisfies the following properties:

1. $B_R(t)$ is Gaussian, that is, for every t > 0, $B_R(t)$ has a normal distribution
2. $B_R(t)$is a self similar process meaning that for any $\xi > 0$, $B_R(\xi, t)$ has the same law as $\xi^R B_R(t)$.
3. It has stationary increments, that is, $B_R(t)$ - $B_R(s) \sim B_R(t-s)$.

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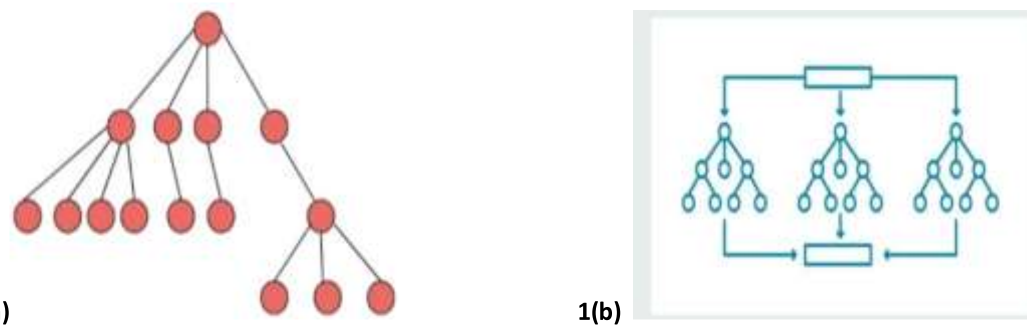
Mandlbrov (1963) [9] studied the Fractional Brownian motion (FBM) and discovered many of its properties.
Fractional Brownian motion finds application in pricing financial derivatives, which are instruments whose price depends on or is derived from the value of another asset, often a stock. The concept of financial derivatives is not new. While there is some historical debate about the exact date of their creation, it is widely accepted that the first attempt at modern derivative pricing began with the work of Charles Castelly (Hui, 2012) [10].
The Black-Scholes option pricing equation, when modeled by fractional Brownian motion, involves replacing the standard Brownian motion in the classical Black-Scholes equation with fractional Brownian motion, which includes the Hurst exponent, R. The Hurst exponent, denoted by R, is a statistical measure used to classify time series. Its value ranges between 0 and 1
The binomial formula, as described by Cox et al. (1979) [11], is a valuable tool for calculating the price of a call option. It is well established that the price computed using the binomial formula converges to the price determined by the Black-Scholes formula, as the number of periods (nn) approaches infinity, as demonstrated by Black & Scholes (1973) [12]
Osu & Duruojinkeya (2023) [13] , on the other hand, proposed a formula for estimating the expected returns of options and stock based on their risk characteristics. In their work, they applied the principles of the binomial method of option pricing and tree theory to calculate the fair value of options. At each node of the tree, they considered two possible outcomes: an increase in the price of the underlying asset and a decrease in the price of the underlying asset.
Building upon this model, our objective is to create a tree pricing model where the underlying asset price dynamics follow Itô-Mckean skew fractional Brownian motion. To determine the worth profile at each node of the tree, we simplify the Black-Scholes option pricing equation, which is modeled by fractional Brownian motion, into a one-dimensional heat equation. We then solve this equation using the Laplace transform method.
Tree theory finds application in various mathematical areas and serves as an intriguing component of combinatorial set theory.
Definition 1.1: Mathematically a tree is a partially ordered set $T =< T_1 \leq T >$ such that for every $X \epsilon T$, the set $x = \{y \in T: y <_T x\}$is well-ordered by $\leq T$.

It is customary to represent trees n pictures from as illustrated below;



1(a) 1(b)

Fig.1: General Tree

Figure 1 illustrates trees with n vertices and $n-1$ edges. The watering in figure 1(a) symbolizes government policies, which, when favorable for investment, can lead to a garden as shown in figure 1(b). This analogy suggests that investors' worth profiles can flourish like trees planted by rivers of water, as mentioned in Psalms 1 verse 3, Osu et al (2024) [13].
The binomial tree method works by initially using the formula for a single-period call option, which can then be expanded to a two-period call option and further to an n–period call option. To create an n–period look-alike option using the binomial tree pricing formula, we divide the effective period T of the option into small

intervals of Δt. At each interval Δt, the stock price changes from S to either S_u (for an upward movement) or S_d (for a downward movement). If the probability of an upward price movement is P, then the probability of a downward movement is $1 - P$.

The formula is derived using a risk-neutral pricing principle because the rate of change in the underlying asset follows a normal distribution. The binomial tree option pricing model has become a standard pricing method for major stock exchanges worldwide (Osu & Duruojinkeya, 2023) [14].

A pricing tree enhances the Black-Scholes (BS) model by considering multiple potential future price paths for the underlying asset, instead of assuming a single deterministic path as the BS model does. This enables a more precise pricing of options, especially in scenarios with substantial uncertainty or volatility in the underlying asset's price. By integrating multiple price paths, a pricing tree can offer a more realistic valuation of option prices and accommodate factors such as mean reversion, jumps, and other forms of price dynamics that the BS model may not fully encompass.

We also have the Conversational BS (ConvoBS) model, which like any other AI model, has several limitations that can impact its performance in various contexts:

1. **Context Dependency:** It might struggle to maintain context over long conversations or understand nuances in context switches.
2. **Common Sense Reasoning:** It may lack comprehensive understanding of common sense knowledge, leading to occasional nonsensical responses.
3. **Emotional Intelligence:** While it can generate responses, it might not always recognize or appropriately respond to emotional cues in a conversation.
4. **Limited Creativity:** It may not consistently produce creative or original responses, relying instead on patterns learned from the training data.
5. **Sensitive Topics Handling:** It may not handle sensitive or controversial topics appropriately, potentially generating offensive or biased responses.
6. **Data Bias:** It may inadvertently perpetuate biases present in the training data, leading to biased or unfair responses.
7. **Complex Reasoning:** It may struggle with complex reasoning tasks or questions requiring deep understanding or critical thinking.
8. **Domain Specificity:** It may not perform well in specialized domains outside its training data.
9. **Open-ended Questions:** It may provide irrelevant or unhelpful responses to open-ended questions that require subjective interpretation or personal opinion.
10. **Ethical Concerns:** It may generate inappropriate or harmful content, raising ethical concerns about its use and deployment.

Despite these limitations, the Black-Scholes (BS) model, developed several decades ago, still has its merits in certain contexts:

1. **Simplicity:** The BS model is relatively straightforward and easy to implement, making it accessible for teaching purposes and for basic option pricing needs.
2. **Speed:** Since it's a closed-form solution, the BS model can be computed quickly compared to some modern computational models, which may require more complex algorithms and longer processing times.
3. **Market Liquidity Assumption:** The BS model assumes continuous trading and perfectly liquid markets, which might be a reasonable approximation for certain highly traded assets in certain conditions.
4. **Historical Context:** Understanding the BS model provides a foundation for understanding more advanced option pricing models, as it was one of the

The task is to determine the value of a stock at each node of the tree depicted in Figure 1. It is noteworthy that each node satisfies a polar form of a diffusion equation in a spherical coordinate system, as expressed by (Datta & Pal, 2018)[15]:

$$\frac{\partial I}{\partial t} = \frac{k}{s^2}\frac{\partial}{\partial r}\left(s^2\frac{\partial I}{\partial r}\right) + \frac{1}{s^2\sin\theta}\frac{\partial I}{\partial\theta}\left(\sin\theta\frac{\partial I}{\partial\theta}\right) + \frac{1}{s^2\sin^2\theta}\frac{d^2 I}{d\emptyset^2}. \qquad (1)$$

where; $I =$ the worth of the stock, $k =$ diffusion coefficient, $(S,\theta,\emptyset) =$ point in spherical coordinate and S is the stock price.

In the radial direction, one-dimension form of (1) can be written as:

$$\frac{\partial I}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 I}{\partial S^2} - r(t)S\frac{\partial I}{\partial S} + r(t)I = 0, \qquad (2)$$

with $I(0,t) = 0$, $I(S,t) \sim S \text{ as } S \to \infty$, where $I =, I(S,t)$ = European option prices; S =Asset price, t = Time, r =Risk-free rate, σ =Volatility.

Definition 1 The Riemann-Liouville fractional integral of $\mathbb{f}$ with order μ is defined by

$$\mathbb{I}^{\mu}\mathbb{f}(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-\tau)^{\mu-1}\mathbb{f}(\tau)\,d\tau \qquad (3)$$

Definition 2 The Riemann-Liouville fractional derivative of $\mathbb{f}$ with order μ is defined by

$$D_x^{\mu}\mathbb{f}(x) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x (x-\tau)^{m-\mu-1}\mathbb{f}(\tau)d\tau\backslash \qquad (4)$$

$${}_k^C D_x^{\mu}\mathbb{f}(x) = \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x (x\tau)^{m-\mu-1}\mathbb{f}(\tau)d\tau \qquad (5)$$

Definition 4 The Mittag-Leffler is defined as

$$E_{\xi} = \sum_{K=0}^{\infty}\frac{z^k}{\Gamma(\xi k+1)}, \xi > 0, z \in C, k = 0,1,\ldots \qquad (6)$$

II. FRACTIONAL OPTION PRICING MODEL

The Fractional Black-Scholes Model improves upon the conventional Black-Scholes Model by incorporating fractional calculus, which enables a more accurate representation of stock price movements over time, particularly in the presence of long memory or persistent behaviors. This model accounts for non-integer variations in asset returns, which the original model overlooks, leading to more precise pricing and risk management strategies, especially for assets with persistent volatility clustering and non-Gaussian characteristics.

Based on a replicating portfolio that ensures no arbitrage opportunities are allowed, a fractional Black-Scholes option can be derived. In the following, we state:

Proposition 2.1: *Let a generic payoff function G(t) = I (S , t). Then the partial differential equation associated with the price of the derivative on the stock price is*

$$\frac{\partial i}{\partial t} + R\sigma^2 S^2 t^{2R-1}\frac{\partial^2 i}{\partial S^2} + rS\frac{\partial i}{\partial S} - ri = 0, \qquad S > 0,\ t > 0, \qquad (7)$$

$$R \in (0,1), R \neq \frac{1}{2}.$$

Where i is the call option price,t is the time to maturity, H is the Hurst exponent, σ is the volatility , S is the stock price and r is the discount rate.

Proof: The stock price S follows the fractional Brownian motion process

$$dS = \mu S dt + \sigma S dB_H(t). \qquad (8)$$

The wealth of an investor X_t follows a diffusion process given by

$$dX = \varsigma dS + r(X - \varsigma S)dt. \qquad (9).$$

Putting equation (8) into equation (9) yields

$$dX - \{rX + \varsigma S(\mu - r)\}dt - \varsigma S\sigma dB_R(t) = 0. \qquad (10)$$

Where μ –r is the risk premium. Suppose that the value of this claim at time t is given by

$$G(t) = i(S,t), S = S_t. \qquad (11)$$

Applying the Ito's formula for fractional Brownian motion on equation (11), we have

$$dG = \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial S}dS + Rt^{2R-1}\frac{\partial^2 I}{\partial S^2}(dS)^2. \qquad (12)$$

Substituting (8) in (12), we have

$$dG = \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial S}[\mu S dt + \sigma S dB_R(t)] + Rt^{2R-1}\frac{\partial^2 I}{\partial S^2}[\mu S dt + \sigma S dB_R(t)]^2. \qquad (13)$$

$$\Rightarrow dG = \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial S}[\mu S dt + \sigma S dB_R(t)] +$$
$$Rt^{2R-1}\frac{\partial^2 I}{\partial S^2}[\mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dt dB_R(t) + \sigma^2 S^2 (dB_R(t))^2]. \quad (14)$$

Multiplication rule implies:

$$(dt)^2 = 0;\ (dB_R(t))^2 = dt$$

Thus (14) reduces to

$$dG = \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial S}[\mu S dt + \sigma S dB_R(t)] + Rt^{2R-1}\sigma^2 S^2 \frac{\partial^2 I}{\partial S^2}dt. \quad (15)$$

Collecting like terms we have

$$dG = \left[\frac{\partial I}{\partial t} + \mu S\frac{\partial I}{\partial S} + Rt^{2R-1}\sigma^2 S^2\frac{\partial^2 I}{\partial S^2}\right]dt + \sigma S\frac{\partial I}{\partial S}\, dB_R(t) \quad (16)$$

Using (11) we have;

$$dI = \left[\frac{\partial I}{\partial t} + \mu S\frac{\partial I}{\partial S} + Rt^{2R-1}\sigma^2 S^2\frac{\partial^2 I}{\partial S^2}\right]dt + \sigma S\frac{\partial I}{\partial S}\, dB_R(t). \quad (17)$$

Thus, equating coefficients, we have

$$\frac{\partial I}{\partial t} + \mu S\frac{\partial I}{\partial S}\ R\sigma^2 S^2 t^{2R-1}\frac{\partial^2 I}{\partial S^2} = rI + \Lambda_t S(\mu - r) \quad (18)$$

and

$$\sigma S\frac{\partial I}{\partial S} = \varsigma_t \sigma S \quad (19)$$

or

$$\varsigma_t = \frac{\partial I}{\partial S}. \quad (20)$$

Substituting equation (20) into (18), we have

$$\frac{\partial I}{\partial t} + \mu S\frac{\partial I}{\partial S}\ R\sigma^2 S^2 t^{2R-1}\frac{\partial^2 I}{\partial S^2} = rI + S\mu\frac{\partial I}{\partial S} - Sr\frac{\partial I}{\partial S} \quad (21)$$

This implies equation (2) as required.

III. The Solution of Equation (2) using Laplace transforms method

To derive the formula for the worth profile at each node of the tree, we begin by simplifying the Black-Scholes option pricing equation, which is modelled by fractional Brownian motion, into a one-dimensional heat equation. We then solve this equation using the Laplace transform method. Subsequently, we state:

Proposition 3.1: *Let equation (7) be given by*

$$\frac{\partial I}{\partial t} + Rt^{2R-1}S^2\sigma^2\frac{\partial^2 I}{\partial S^2} + rS\frac{\partial I}{\partial S} - rI = 0,\ S > 0, t > 0;\ I(0,t) = 0, I(S,t) \sim S \text{ as } S \to \infty,$$
$$I(S,t) = max\{|\mathrm{S} - \mathrm{K}|, 0\} \quad (22)$$

Then (22) can be reduced to one-dimensional heat equation of the form

$$\frac{\partial u}{\partial \tau} = p\frac{\partial^2 u}{\partial x^2}. \quad (23)$$

Proof: Set

$$\tau = \frac{\sigma^2(T-t)}{2}, x = \ln(S/K) \quad (24)$$

and

$$I(S,t) = KI(x,\tau). \quad (25)$$

Differentiating (24) and (25), we have

$$\frac{\partial I}{\partial t} = K\frac{\partial I}{\partial \tau}.\frac{\partial \tau}{\partial t} = \left(K\frac{\partial I}{\partial \tau}\right)\left(-\frac{\sigma^2}{2}\right), \quad (26)$$

$$\frac{\partial I}{\partial S} = K\frac{\partial i}{\partial x}.\frac{\partial x}{\partial S} = K\frac{\partial i}{\partial x}\left(\frac{1}{S}\right) = \frac{K}{S}\frac{\partial i}{\partial x}, \quad (27)$$

$$\frac{\partial^2 I}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial I}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial i}{\partial x}\right) = \frac{K}{S}\left(\frac{\partial}{\partial S}\frac{\partial i}{\partial x}\right) + \frac{\partial i}{\partial x}\left(\frac{\partial}{\partial S}\frac{K}{S}\right)$$
$$= \frac{K}{S}\left(\frac{\partial}{\partial S}\frac{\partial i}{\partial x}\right) + \frac{\partial i}{\partial x}\left(\frac{-K}{S^2}\right) = \frac{K}{S}\left[\frac{\partial}{\partial x}\left(\frac{\partial i}{\partial x}\right)\frac{dx}{dS}\right] - \frac{K}{S^2}\frac{\partial i}{\partial x} = \frac{K}{S}\frac{\partial^2 i}{\partial x^2}\left(\frac{1}{S}\right) - \frac{K}{S^2}\frac{\partial i}{\partial x}.$$

Therefore

$$\frac{\partial^2 I}{\partial S^2} = -\frac{K}{S^2}\frac{\partial i}{\partial x} + \frac{K}{S^2}\frac{\partial^2 i}{\partial x^2}. \quad (28)$$

The terminal condition is

$$I(S,T) = max\{|S-K|,0\}=\max\{|Ke^x - K|,0\}.$$

Let

$$I(S,T) = Kv(x,0)$$

$$\Rightarrow i(x,0) = \max\{|e^x - 1|,0\} \quad (29)$$

Substitute (26), (27) and (28) in (22) and get

$$\left(K\frac{\partial I}{\partial \tau}\right)\left(\frac{\sigma^2}{2}\right) + R\left(T - \frac{2\tau}{\sigma^2}\right)^{2R-1} S^2\sigma^2\left(-\frac{K}{S^2}\frac{\partial i}{\partial x} + \frac{K}{S^2}\frac{\partial^2 i}{\partial x^2}\right) + rS\left(\frac{K}{S}\frac{\partial i}{\partial x}\right) - rKi = 0. \quad (30)$$

Let (the correlation coefficient)

$$m = \frac{2\tau}{\sigma^2}, \quad (31)$$

then we have

$$-\frac{\sigma^2}{2}\frac{\partial i}{\partial \tau} + R(T-m)^{2R-1}S^2\sigma^2\left(-\frac{1}{S^2}\frac{\partial i}{\partial x} + \frac{1}{S^2}\frac{\partial^2 i}{\partial x^2}\right) + rS\left(\frac{1}{S}\frac{\partial i}{\partial x}\right) - rv = 0\,, \quad (32)$$

and

$$-\frac{\sigma^2}{2}\frac{\partial i}{\partial \tau} + R(T-m)^{2R-1}\sigma^2\frac{\partial i}{\partial x} + R(T-m)^{2R-1}\sigma^2\frac{\partial^2 i}{\partial x^2} + r\frac{\partial i}{\partial x} - ri$$

$$= -\frac{\sigma^2}{2}\frac{\partial i}{\partial \tau} + R(T-m)^{2R-1}\sigma^2\frac{\partial i}{\partial x} - r\frac{\partial i}{\partial x} - R(T-m)^{2R-1}\sigma^2\frac{\partial^2 i}{\partial x^2} + ri$$

$$= \frac{\sigma^2}{2}\frac{\partial i}{\partial \tau} + [R(T-m)^{2R-1}\sigma^2 - r]\frac{\partial i}{\partial x} - R(T-m)^{2R-1}\sigma^2\frac{\partial^2 i}{\partial x^2} + ri$$

$$= \frac{\partial i}{\partial \tau} + \left[2R(T-m)^{2R-1} - \frac{2r}{\sigma^2}\right]\frac{\partial i}{\partial x} - 2R(T-m)^{2R-1}\frac{\partial^2 i}{\partial x^2} + \frac{2ri}{\sigma^2}. \quad (33)$$

Let

$$p = 2R(T-m)^{2R-1} \quad (34)$$

Then one has

$$\frac{\partial i}{\partial \tau} + (p-q)\frac{\partial i}{\partial x} - p\frac{\partial^2 i}{\partial x^2} + qi = 0 \quad (35)$$

or

$$\frac{\partial i}{\partial \tau} = p\frac{\partial^2 i}{\partial x^2} + (q-p)\frac{\partial i}{\partial x} - qi. \quad (36)$$

Furthermore let

$$i(x,\tau) = e^{\xi x + \psi\tau}u(x,\tau). \quad (37)$$

Then using the product rule, we have

$$i_\tau = \psi e^{\xi x+\psi\tau}u + e^{\xi x+\psi\tau}u_\tau,$$

$$v_x = \xi e^{\xi x+\psi\tau}u + e^{\xi x+\psi\tau}u_x$$

and

$$v_{xx} = \xi^2 e^{\xi x+\psi\tau}u + 2\xi e^{\xi x+\psi\tau}u_x + e^{\xi x+\psi\tau}u_{xx}.$$

where i_τ and i_x stand for the first partial derivative of i with respect to τ and x respectively. $u_\tau$andu_x stand for the first partial derivative of u with respect to τ and x respectively. i_{xx}andu_{xx} stand for the second partial derivative of i and u with respect to x .

Substituting into (37), one gets

$$\psi e^{\xi x+\psi\tau}u + e^{\xi x+\psi\tau}u_\tau$$

$$=p\left[\xi^2 e^{\xi x+\psi\tau}u + 2\xi e^{\xi x+\psi\tau}u_x + e^{\xi x+\psi\tau}u_{xx}\right] + (q-p)\left[\xi e^{\xi x+\psi\tau}u + e^{\xi x+\psi\tau}u_x\right]-qe^{\xi x+\psi\tau}u.$$

Simplifying one has $\psi u + u_\tau = p[\xi^2 u + 2\xi u_x + u_{xx}] + (q-p)(\xi u + u_x) - qu,$

$$u_\tau = pu_{xx} + [2\xi p + (q-p)]u_x + [\xi^2 p + (q-p)\xi - q - \psi]u. \quad (38)$$

Choose

$$\xi = \frac{p-q}{2p} \text{and } \psi = \frac{-(p+q)^2}{4p}. \quad (39)$$

Thus (38) is reduced to

$$u_\tau = pu_{xx}. \quad (40)$$

Equation (40) is a one dimensional heat equation. One can solve (40) using Laplace

transforms method.

Theorem 3.1: Let the worthprofile $u(x,t)$ in each nod (as in figure 1) obey the heat diffusion equation of equation (40).

$$\frac{\partial}{\partial t}u(x,t) = p\frac{\partial^2}{\partial x^2}u(x,t), \quad (41)$$

such that initial worth distribution $u(x) = u_0$. At $t = 0$, the worth at the half end is changed instantaneously to $u(0,t) = 0$ and kept at the worth profile for all $t > 0$,Then

(i) $i(x,\tau) = e^{\xi x+\psi\tau}\frac{x}{2\sqrt{\pi 2R(T-m)^{2R-1}t^3}}exp\left\{-\left(\frac{x^2}{4p2R(T-m)^{2R-1}t}+t\right)\right\}$, if one assumes a solution for <0 , and

(ii). $i(x,\tau) = u_0 e^{\xi x+\psi\tau}erfc\left(\frac{x}{2\sqrt{2R(T-m)^{2R-1}t}}\right)$ if > 0.

Proof:

The Laplace of the left-hand side of (41) gives;

$$L\left(\frac{\partial}{\partial t}i(x,t)\right) = s\phi(x,s) - i(x,0) \; = s\phi(x,s) - u_0 \,. \quad (42)$$

And of the right-hand side:

$$L\left(\frac{\partial^2}{\partial x^2}i(x,t)\right) = \frac{\partial^2}{\partial x^2}\phi(x,s),$$

which gives

$$s\phi(x,s) - T_0 = \frac{\partial^2}{\partial x^2}\phi(x,s) \rightarrow \frac{\partial^2}{\partial x^2}\phi(x,s) - s\phi(x,s) = u_0$$

or

$$\frac{\partial^2\phi}{\partial x^2} - \frac{s}{p}\phi = u_0. \quad (43)$$

Consider

$$\phi'' - \frac{s}{p}\phi = u_0. \quad (44)$$

Then this has the particular integral

$$\phi'' - \frac{s}{p}\phi = 0,$$

with auxiliary equation

$$n^2 - \frac{s}{p} = 0,$$

with

$$n = \pm\sqrt{\frac{s}{p}}.$$

And from here this is solved by considering cases for $s = 0, s > 0$. For $s < 0, m$isimaginary and the solution for

$$\phi = c_1\cos\left(\sqrt{\frac{s}{p}}x\right) + c_2\sin\left(\sqrt{\frac{s}{p}}x\right) = e^{\sqrt{\frac{s}{p}}x}, c_1 = c_2 = 1. \quad (45)$$

There should not be a need to consider $s < 0$, as the Laplace variable is usually assumed to be > 0 by definition and one has not considered any separation of variables. However, if one considers equation (45) as a solution then the inverse transform will give

$$u(x,t) = \frac{x}{2\sqrt{\pi p t^3}}exp\left\{-\left(\frac{x^2}{4pt}+t\right)\right\}. \quad (46)$$

Combining (46) and (37) gives the worth profile in this case as (using (34));

$$i(x,\tau) = e^{\xi x+\psi\tau}\frac{x}{2\sqrt{\pi 2R(T-m)^{2R-1}t^3}}exp\left\{-\left(\frac{x^2}{4p2R(T-m)^{2R-1}t}+t\right)\right\} \quad (47)$$

Now set

$$L_t(n(x,t)) = U(x,s),$$
$$L(u'') = L(\dot{u}) \rightarrow U''(x,s) = \frac{s}{4}U(s,s) - \frac{1}{4}u(x,0).$$

Then (with a sign error),one gets;

$$\phi_{xx} - \frac{s}{p}\phi + u_0 = 0. \quad (48)$$

For each fixed s, this represents a constant-coefficient second-order linear ordinary differential equation (ODE) in x. The general solution is obtained by summing the general solution to the homogeneous equation. So for $s > 0$ the solution is given as a sum of $c_1 e^{\sqrt{\frac{s}{p}}x} + c_2 e^{-\sqrt{\frac{s}{p}}x}$ and by inspection $\phi_p(x) = \frac{u_0}{s/p}$ is a solution of the inhomogeneous equation.

Using the initial-value theorem for the Laplace transform $\left(f(0) = \lim_{s\to\infty} sF(s)\right)$ to show that $c_1 = 0$. The boundary condition $\phi(0,t) = 0$ implies $\phi(0,s) = 0$ for all $s > 0$, which then implies $c_2 = -\frac{u_0}{s/p}$. Altogether from here we obtain the Laplace transform is

$$\phi(x,s) = \frac{-u_0 p}{s}\left[-1 + e^{-\sqrt{\frac{s}{p}}x}\right]. \quad (49)$$

Next, we invert this Laplace transform. The second term simply results in a unit step function. However, the inverse Laplace transform of the first term cannot be expressed in terms of elementary functions. Nonetheless, we can express it using the rule:

$$F(s) = sL\left[\int_0^t f(\tau)d\tau\right](s), \quad (50)$$

which gives

$$\phi(x,t) = -\left(T_0 \int_0^t L^{-1}\left[e^{-\sqrt{\frac{s}{p}}x}\right](\tau)d\tau - u_0 u(t)\right). \quad (51)$$

where $u(t)$ is the unit step function. The inverse Laplace transform in the integral gives

$$u(x,t) = L^{-1}\left[e^{-\sqrt{\frac{s}{p}}x}\right](\tau) = \frac{xe^{-\frac{x^2}{4\tau}}}{2\sqrt{p\pi}\tau^{\frac{3}{2}}}, \quad (52)$$

Making the change of variable $\eta = \frac{x}{2\sqrt{\tau}}$ or equivalently $= \frac{x^2}{4\eta^2}$, $d\tau = \frac{x^2}{2\eta^3}d\eta$ gives

$$\int_0^{-1} L^{-1}\left[e^{-\sqrt{\frac{s}{p}}xx}\right](\tau)d\tau = \frac{2}{\sqrt{p\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-\eta^2}d\eta = erfc\left(\frac{x}{2\sqrt{pt}}\right). \quad (53)$$

Where $erfc$ denotes the complementary error function. Therefore, the solution comes out to

$$u(x,t) = u_0 erfc\left(\frac{x}{2\sqrt{pt}}\right) - u_0 u(t). \quad (54)$$

To check that this is consistent with the initial and boundary conditions at $t = 0$,
$erfc\left(\frac{x}{\sqrt{pt}}\right) = 0$for all $x > 0$, so $\phi(x,0) = u_0$ for all $x > 0$,

While at $x = 0$, one can explicitly calculate the value of $erfc(\frac{s}{p})$ (It is half of the famous Gaussian Integral) to find that $u(0,t) = -u_0 + u_0 = 0$.

Therefore

$$u(x,t) = u_0 erfc\left(\frac{x}{2\sqrt{pt}}\right) \text{ as } u(t) \uparrow 0. \quad (55)$$

Combining (55) and (37) gives the worth profile in this case as (using (34));

$$i(x,\tau) = u_0 e^{\xi x + \psi\tau} erfc\left(x[2R(T-m)^{2R-1}t]^{-\frac{1}{2}}\right). \quad (56)$$

IV. Sensitivity Analysis for the Skewed Profile and Some numerical examples

To carry out sensitivity analysis of the worth of a stock using a random pricing tree and the Time-Regular Long Wave (TRLW) equation, one would typically follow these steps below:

1. Identify Sensitivity Parameters: Determine which parameters or factors are most influential in the valuation process. These could include parameters related to the underlying asset (e.g., volatility, dividend yield), market conditions (e.g., interest rates, liquidity), or model assumptions (e.g., time regularity parameter in the TRLW equation).
2. Define Parameter Ranges: Specify a range of values for each sensitivity parameter over which you want to conduct the analysis. This could involve varying parameters individually or in combination to assess their impact on option prices.
3. Generate Scenarios: Use the random pricing tree approach to generate a set of scenarios reflecting different combinations of parameter values within the defined ranges. This may involve simulating multiple paths for the underlying asset price based on the specified parameter values.
4. Calculate Option Prices: For each scenario, use the TRLW equation to calculate the option prices corresponding to the generated asset price paths. This may require solving the pricing equation numerically or using approximation techniques.
5. Analyze Results: Analyze the variation in option prices across the different scenarios to assess the sensitivity of the option values to changes in the input parameters. This may involve calculating sensitivity measures such as delta, gamma, vega, and theta to quantify the impact of each parameter on option prices.
6. Identify Key Drivers: Identify which parameters have the most significant impact on option prices and how their effects vary across different scenarios. This can help in understanding the key drivers of option valuation and informing risk management decisions.
7. Conduct Scenario Analysis: In addition to individual parameter sensitivity analysis, consider conducting scenario analysis to assess the impact of specific market events or economic conditions on option prices. This could involve simulating extreme market scenarios or stress testing the model under different assumptions.
8. Validate Results: Validate the results of the sensitivity analysis by comparing them with observed market data or benchmark values obtained from alternative pricing models. Ensure that the sensitivity analysis accurately reflects the behavior of option prices under different parameter values and market conditions.

By following these steps, you can gain insights into the sensitivity of option prices to changes in key parameters and assess the robustness of the valuation framework based on random pricing trees and the TRLW equation.

A levy process $(i(\tau))_{\tau\geq 0}$ with a nonnegative increment is called a subordinator. The Laplace transform of $i(\tau)$ has the form $\mathbb{E}\left[e^{-\xi i(\tau)}\right] = e^{-r\varphi(i)}$, ≥ 0 with the Laplace exponent $\varphi(i)$ given by

$$\varphi(z) = \psi z + \int_0^\infty (1 - e^{-xz}) d\omega(x).$$

For any complex z with the $Re\, z \geq 0$,where$\psi \geq 0$ is a drift parameter and ω is a measure satisfying $\int_0^\infty \min\{1, x\} d\omega(x) < +\infty$ which is called Levy measure of $(i(\tau))_{\tau\geq 0}$.
For $\xi \in (0,1)$, the Levy measure of $i_\xi(t)$ is absolutely continuous with respect to the Lebesque measure on $(0, +\infty)$ with the density function (as in the set of equations (3-6);

$$h_\xi(x) = \frac{\xi}{\Gamma(1-\xi)x^{\xi+1}}, x > 0. \qquad (57)$$

It therefore suggests that $[i_\omega(t)]_{\tau\geq 0}$ is a stochastic process with Laplace transform;

$$\mathbb{E}e^{-\mu i_\gamma(\tau)} = e^{-\tau\varphi_\gamma(i)}, \qquad (58)$$

with a subordinator, where $\varphi_\gamma(u) \geq \int_0^\tau i^x d\gamma(x)$ and γ is a Borel probability measure on $(0,1)$.
Park & Nguyen (2023) [16] investigated the Black-Litterman (BL) asset allocation model under the assumption of a hidden truncation skew-normal distribution. They demonstrated that when returns are assumed to follow this skew-normal distribution, the posterior returns, after incorporating views, also follow a skew-normal distribution.

Let $i = \{i_t,\ t \geq 0\}$ be a standard Brownian Motion generating a stochastic basis $[(\Omega, \mathbb{F} = (\mathfrak{T}_t = \sigma(v_\xi, \xi \leq t): t \geq 0, P)$(Li, 2019) [6].

Let $\xi \in (0,1)$ and set

$$\mathcal{A}_t^{(\xi)} = \int_0^\tau \xi^2 I_{\{i_s \geq 0\}} + (1-\xi)^2 I_{\{i_s \geq 0\}} ds,$$
$$\tau_{it}^{(\xi)} = \inf\left\{s \geq 0: \mathcal{A}_s^{(\xi)} > t\right\},\ t \geq 0,$$
$$i_t^{(\xi)} = \varphi_\xi\left(i_{\tau_t^{(\xi)}}\right), t > 0, i_0^{(\xi)} > 0.$$

Where $\varphi_\xi(x) = \xi x I_{\{x \geq 0\}} + (1-\xi) x I_{\{x<0\}}, x \in R$. Then the process $\mathbb{v}^{(\xi)} = \{i_t^{(\xi)}, t \geq 0\}$ is called a skewed Brownian Motion (SBM) with parameter ξ. For every $t \geq 0$, the density $f_t^\xi(x), x \in R$ of $i_t^\xi(x)$ is given by;

$$i_t^\xi(x) = \begin{cases} \xi\left[u_0 e^{\xi x+\psi\tau} erfc\left(\frac{x}{2\sqrt{2R(T-m)^{2R-1}t}}\right)\right], & \text{if } x \geq 0 \\ (1-\xi)\left[e^{\xi x+\psi\tau}\frac{x}{2\sqrt{\pi 2R(T-m)^{2R-1}t^3}} exp\left\{-\left(\frac{x^2}{4p2R(T-m)^{2R-1}t}+t\right)\right\},\ if\ x<0\right] \end{cases} \quad (59)$$

In order to find the worth of a stock at each point of tree, equation (59) can be written in polar coordinate system as;

$$i_t^\xi(\omega) = \begin{cases} \xi\left[u_0 e^{\xi x+\psi\tau} erfc\left(\frac{\omega\cos\theta}{2\sqrt{2R(T-m)^{2R-1}t}}\right)\right], & \text{if } \omega \geq 0 \\ (1-\xi)\left[e^{\xi x+\psi\tau}\frac{\omega\cos\theta}{2\sqrt{\pi 2R(T-m)^{2R-1}t^3}} exp\left\{-\left(\frac{(\omega\cos\theta)^2}{4p2R(T-m)^{2R-1}t}+t\right)\right\},\ if\ \omega<0\right]' \end{cases} \quad (60)$$

where $x = \omega\cos\theta$.

Equation (60) is the worth profile at each node with the possibilities of up or down nodes of the tree as in figure 1b.

Now assume an expected volume of portfolio at each node of a tree (see figure 1) given values τ, m, p, q, ξ and ψ as calculated from equations (24), (31), (34) and (39) respectively. Then the worth distribution in the volume of portfolio at time, t, at each node is as in figures 2 and 3 below probability up or down

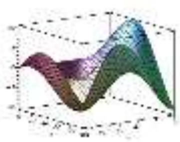


Figure 2: Worth profiles of equation (47),for the down nodes of the tree of equation (59). Substituting different Values of the Parameters p, q, t, ξ, ψ, r for the 3D Graphs, we calculate the expected value $i(x,t)$ with current price of a stock $\omega_0 = \$120$ and the expiry date is 363day, the size of the up move $\xi = 0.131$ and the risk free rate $r = 0.09$. We use a binomial tree, to determine the current worth of the option. Herein, we determine the probability for stock price upturn uniquely as $P_\xi = \frac{1}{2} + \frac{\left|\xi - \psi^2/2\right|}{\psi}\sqrt{\Delta t}$, $P_\xi = 0.521, P_\psi = 0.479$, $\psi = 0.242$, where P_ξ, $P_\psi = 1 - P_\xi$ are the probabilities of up or down nodes of the tree as in figure 1b and $\Delta t = 0.05$.

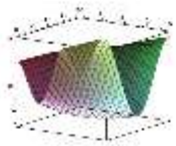


Figure 3: Worth profiles of equation (56), for up node of the tree of equation (60). Substituting different Values of the Parameters p, q, t, ξ, ψ, r for the 3D Graphs, we calculate the expected value $i(x, t)$ with current price of a stock $\omega_0 = \$120$ and the expiry date is 363day, the size of the up move $\xi = 0.131$ and the risk free rate $r = 0.09$. We use a binomial tree, to determine the current worth of the option. Herein, we determine the probability for stock price upturn uniquely as $P_\xi = \frac{1}{2} + \frac{\left|\xi - \psi^2/2\right|}{\psi}\sqrt{\Delta t} P_\xi = 0.521, P_\psi = 0.479$, $\psi = 0.242$, where P_ξ, $P_\psi = 1 - P_\xi$ are the probabilities of up or down nodes of the tree as in figure 1b and $\Delta t = 0.05$.

Figures 4(a,b) below is the relationship between the optimal worth control strategy and control coefficient, P and control policies, q with varying risk over time, for all other parameters remain fixed. It is observed that the investment portfolio in the risky asset is positive.

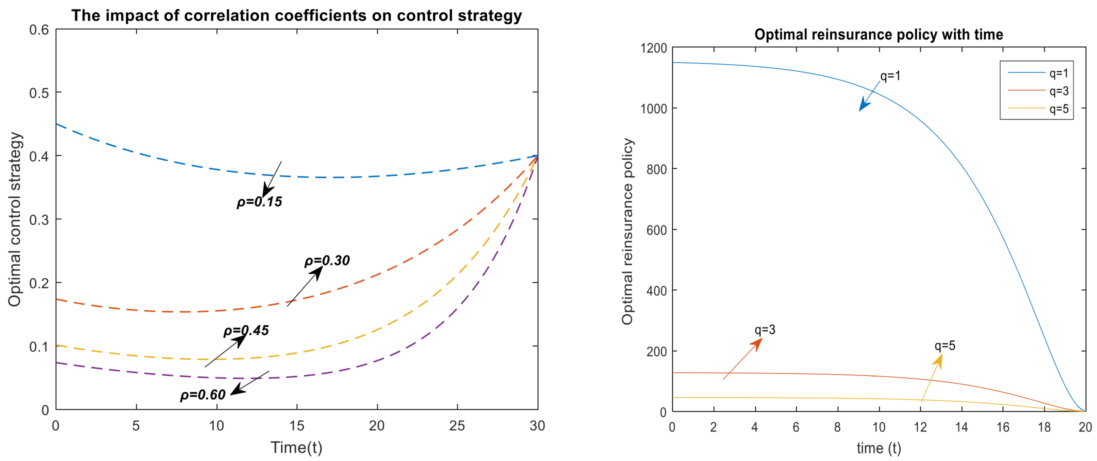


Figure 4(a, b): The effect of control coefficient, P and control policies, q on the Spatial profile of worth concentration after 365, 1000 and 3000days.

V. Conclusion

In this study, the Laplace transform proved instrumental in establishing the existence of a unique solution for the heat partial differential equation incorporating stochastic variables. The transformed equation was formulated in terms of volatile parameters, which significantly influence the system. These effects were meticulously scrutinized through numerical simulations. The objective of deriving a fractional formula for the profile of worth option payoff, with the goal of analyzing the deposit at each node of a tree within the framework of a binomial tree insurance model, was successfully achieved.

The optimal portfolio derived through this method demonstrates lower risk compared to an optimal portfolio from the classical Black-Scholes (B-S) model. As the expected returns of portfolios increase, the optimal portfolio becomes more negatively skewed. This implies that investors are willing to trade negative skewness for a higher expected return. Additionally, there may be a negative relationship between portfolio volatility and portfolio skewness, indicating that investors might choose between lower volatility and higher skewness. This trade-off suggests that stocks with substantial price declines tend to have increased volatility.

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