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Solving Nonlinear Integral Equations Numerically by Matlab

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Abstract: *We used the numerical mathematical method by Matlab to solve the nonlinear integral equations. We reviewed the different numerical methods for solving the nonlinear integral equations by a new mathematical technique which it called matlab. Our aim is to solve nonlinear integral equations numerically by matlab. Moreover the Adomian decomposition method does not apply here because it depends mainly on the assignment of a zero value. We stress the importance of this study for solving the nonlinear integral equations .Finally we found that the solvingof nonlinear integral equations using numerical methods by Matlab gives the most and best accurate solutions.*

Keywords: *Nonlinear Integral Equations, Volterra Integral Equation, Fredholm Integral Equation, Numerical Method s, Trapezoidal rule, Nyström (Qudrature) method.*

I. Introduction

In general the solution of the nonlinear integral equations isnot unique. However the existence of a unique solution of nonlinear integral equations with specific conditions is possible. As we know there is a close relationship between the differential equations and the integral equations. We will see in the next some classical development of these two systems and the numerical methods ofsolutions by Matlab.In general a nonlinear integralequations is defined as given in the following equation:$u(x) = f(x) + \lambda \int_a^b K(x,t)F\big(u(t)\big)dt$, which it called nonlinear integral equation of the second kind .We mostly use degenerate or separable kernels. A degenerate or a separable kernel is a function that can be expressed as the sum of product of two functions each depends only on one variable.Several analytic and numerical methods have been used to handle thenonlinear integral equations.For each type of equations we select the proper methods that facilitate the computational work.The emphasis in this text be on the use of these methods rather than provingtheoretical concepts of convergence and existence.The concern be on the determination of the solutions $u(x)$ of the nonlinear integral equations.

II. Matlab

Matlab is a very useful piece of software with extensive capabilities for numerical computation and graphing. It offers a powerful programming language, excellent graphics, and a wide range of expert knowledge. MATLAB is published by and a trademark of The Math Works, Inc.

MATLAB®, developed by The MathWorks, Inc., integrates computation, visualization, and programming in a flexible, open environment. It offers engineers, scientists, and mathematicians an intuitive language for expressing problems and their solutions mathematically and graphically. Complex numeric and symbolic problems can be solved in a fraction of the time required with a programming language such as C, Fortran, or Java. [9], [10].

III. Numerical Methods of Nonlinear Integral Equations of the Second Kind

i. Numerical Methods of Nonlinear Volterra Integral Equations of the Second Kind

a.Trapezoidal rule:

Let $a < b \in R$. We divide the interval (a ,b) into subintervals with equal length $h = \frac{b-a}{N}$ .we denote $x_i = a + (i-1)h$, $1 \le i \le N+1$, then the Trapezoidal method reads:

$$\int_a^b f(x)\,dx = h\left[\frac{f(a)+f(b)}{2} + \sum_{i=2}^{N-1} f(x_i)\right] \quad (1)$$

Using the Trapezoidal approximation to solve the Volterra integral equation:

$$u(x) - \lambda \int_a^b k\,(x,t)u(t)dt = f(x) \quad (2)$$

We substitute (1) into (2) with x_i , we get

$$u(x_i) - h\left[\frac{k(x_i\,,a)u(a) + k(x_i\,,x_i)u(a)}{2} + \sum_{j=2}^{i-1} k(x_i\,,x_j)u(x_i)\right] = f(x_i) \quad (3)$$

$$1 \le i \le N+1\ , \quad x_1 = a, x_2\,, \ldots\, x_{N+1}$$

$$= b - h\frac{k(x_i\,,a)}{2}\,u(a) - h\sum_{j=2}^{i-1} k(x_i\,,x_j)\,u(x_j) + \left(1 - h\frac{k(x_i\,,x_i)}{2}\right)u(x_i) = f(x_j)$$

For $i = 1\,, x_1 = a$, theVolterra integral equation (2) is reduced to

$$u(a) = f(a)$$

For $i = 2$, we get

$$-h\,\frac{k(x_2\,,x_1)}{2}u(x_1) + \left(1 - h\frac{k(x_2\,,x_2)}{2}\right)\,u(x_2) = f(x_2)$$

For $i = 3$, we obtain

$$-h\,\frac{k(x_3\,,x_1)}{2}u(x_1) - hk(x_3\,,x_2)u(x_2) + \left(1 - h\frac{k(x_3\,,x_3)}{2}\right)\,u(x_3) = f(x_3)$$

To this end, we obtain the linear system

$$A\bar{a} = B$$

Where the matrix $A = (a_{ij})$, $1 \le i,j \le N+1$with :

$$a_{ij} = 0 \qquad \forall j \le i+1$$

$$a_{ij} = -hk\,(x_i\,,x_j)\,, \quad 2 \le j \le i \le n+1$$

$$a_{ii} = 1 - \frac{h}{2}k(x_i\,,x_i)$$

$$a_{11} = 1$$

$$a_{i1} = -\frac{h}{2}k(x_i\,,x_1)\,, \quad 1 \le i \le n+1$$

$$A = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ a_{N+1,1} & a_{N+1,2} & \cdots & \cdots & a_{N+1,1,N+1} \end{bmatrix}$$

$$B = [f(x_1) = f(a), f(x_2) \dots, f(x_{N+1}) = f(b)]^T \ ,$$

$$\overline{u} = [u(a), u(x_2) \dots, u(x_{N+1})]^T$$

[1] , [2] , [3] , [4] and [5].

ii. Numerical Methods of Nonlinear Fredholm Integral Equations of the Second Kind:

$$f(x) = g(x) + \lambda \int_{\Delta} G\,(x,y) f(y) dy \quad x \in D \qquad (4)$$

$\lambda \neq 0$, $D \subset R^m$,for some $m \geq 1$ where D is a closed and bounded set.

a.**Nyström (Qudrature) method :[6]**

The Nyström method was found to handed approximations based on numerical integration of the integral operator in the equation (4) the solution is found first at the set of quadrature node points and then it is extended to all points in D by means of a special interpolation formula. The numerical is much simpler to implement on a computer, but the error analysis is more sophisticated than for the methods of the preceding two sections. For solving the Fredholm integral equation in (4) by this method.

We use the numerical integration scheme.

$$\int_D h\,(y) dy \approx \sum_{j=1}^{k_n} w_{n,j}\, h(x_{n,j}), \qquad h \in C(D) \qquad (5)$$

With an increasing sequence of values of n. Assuming that the numerical integrals for every $h \in D$ converge to the true integral as $n \to \infty$.

To simplify the notation, we omit the subscript n so that $w_{n,j} \equiv w_j$, $x_{n,j} \equiv x_j$ and some times $k_n \equiv k$, but we understand the presence of n implicitly.

Let the kernel function be continuous for all $x, y \in D$ where D is a closed and bounded set in R^m for some $m \geq 1$. By approximating the integral in (4) using the quadratuere scheme in (5). We obtain a new equation

$$f_n(x) - \lambda \sum_{j=1}^{k_n} w_j\, G(x, x_j) f_n(x_j) = g(x), \quad x \in D \qquad (6)$$

Where its solution $f_n(x)$ is an approximation of the exact solution $f(x)$ to (4).A solution to a functional equation (6) may by obtained if we assign $x_i's$ to x in which $i = 1, \dots k_n$ and $x_i \in D$. In this way , (6) is reduced to the system of equation

$$f_n(x_i) - \lambda \sum_{j=1}^{k_n} w_j\, G(x_i, x_j) f_n(x_i) = g(x_i), \quad i = 1, \dots k_n \qquad (7)$$

Which is a linear system of order k_n .the unknown is a vector.

$$f_n \equiv [f_n(x_1), \dots, f_n(x_q)]^t$$

Each solution $f_n(x)$ of (6) furnishes a solution to (7).merely evaluate $f_n(x)$

At the nod points and $D = diag\,(w_1, w_2, \dots, w_k)$.

It is worth noting that $I - \lambda kD$ may be singular for a chosen quadrature rule (5).

However , under suitable restrictions, we can preserve the non singularity of $I - \lambda kD$ if we decide on a sufficiently accurate (5) in addition , whether quadrature rule is sufficiently accurate or not itself depends on λ , $G(x, y)$, and $g(x)$. [7] , [8].

IV. Solving Nonlinear Integral Equations Using Matlab

i. Nonlinear Volterra Integral Equations:

Example (4.1):

Consider the Volterra integral equation of the second kind

$$u(x) = 2e^{x} - x - 2 + \int_{0}^{x} (x-t)\, u(t)dt. \qquad (8)$$

Equation (8) has the exact solution

$u(x) = xe^{x}$.

We will find an approximate solution to equation (8) by the following numerical method:

ii.The Numerical Realization of Equation (8) using Trapezoidal Rule :

The following algorithm implements theTrapezoidal rule using the Matlab software.

Algorithm (4.2):

1 .input : the number of subdivisions of $[a, b]$

a, b: $[a, b]$is the interval for the solution function

$fcn - f$: the handle of the driver function $f(x)$.

and $fcn - k$: the handle of the kernel function $k(x, t)$

2. Loop=10 this is much more than is usually needed.
3. Calculate $h = (b - a)/n$
4. Calculate $x = linspace\ (a, b, n + 1)$
5. Calculate $f - vec = fcn - f(x)$
6. $set\ u - vec = zeros\ (size(x))$
8. $for\ i = 1{:}n$

$u - vec(i + 1) = u - vec(i)$.the initial estimate for the iteration.

$$k - vec = fcn - k(x(i + 1)\,, x(1{:}n + 1)) * u - vec(1{:}n + 1)$$

for$j = 1{:}Loop$

applying trapezoid rule

$$u - vec(i + 1) = f - vec(i + 1) + h * (sum\big(k - vec(2{:}i)\big) + \cdots$$
$$(k - vec(1) + k - vec(i + 1))/2)$$
$$k - vec(i + 1) = fcn - k\big(x(i + 1), x(i + 1)\big) * u - vec(i + 1)$$

end

end

9. $set\ u = u - vec$
10. output :the numerical solution $u(x)$. And the grid points x at which the solution $u(x)$ is approximated.

Thus we can solve the Volterra integral equation of the second kind (8) by using algorithm 1.1 Table 1.1 shows the exact and numerical results when $n = 20$, and showing the error resulting of using the numerical solution.

Table (1): The Exact and Numerical Solutions for Algorithm (4.2):

X	Analytical solution $u(x) = xe^{x}$	Approximate solution $u_n(x)$	Error $\lvert u - u_n \rvert$
0	0	0	0
0.05	0.052563555	0.052563554	0.02136×10^{-3}
0.1	0.110517092	0.110517091	0.04390×10^{-3}
0.15	0.174275136	0.174275136	0.06775×10^{-3}
0.2	0.244280552	0.244280551	0.09308×10^{-3}
0.25	0.321006354	0.321006354	0.12004×10^{-3}
0.3	0.404957642	0.404957642	0.14880×10^{-3}
0.35	0.496673642	0.496673642	0.17956×10^{-3}
0.4	0.596729879	0.596729879	0.212503×10^{-3}

0.45	0.705740483	0.705740483	0.247840×10^{-3}
0.5	0.824360635	0.824360635	0.285798×10^{-3}
0.55	0.95328916	0.953289159	0.326617×10^{-3}
0.6	1.09327128	1.093271280	0.370555×10^{-3}
0.65	1.245101539	1.245101538	0.417888×10^{-3}
0.7	1.409626895	1.409626895	0.468909×10^{-3}
0.75	1.58775012	1.587750012	0.523935×10^{-3}
0.8	1.780432743	1.780432742	0.583304×10^{-3}
0.85	1.988699824	1.988699824	0.647376×10^{-3}
0.9	2.2136428	2.213642800	0.716541×10^{-3}
0.95	2.456424176	2.456424176	0.791212×10^{-3}
1	2.718281828	2.7182818284	0.871835×10^{-3}

Figure (1): Shows both the Exact and the Numerical Solutions with $n = 20$.

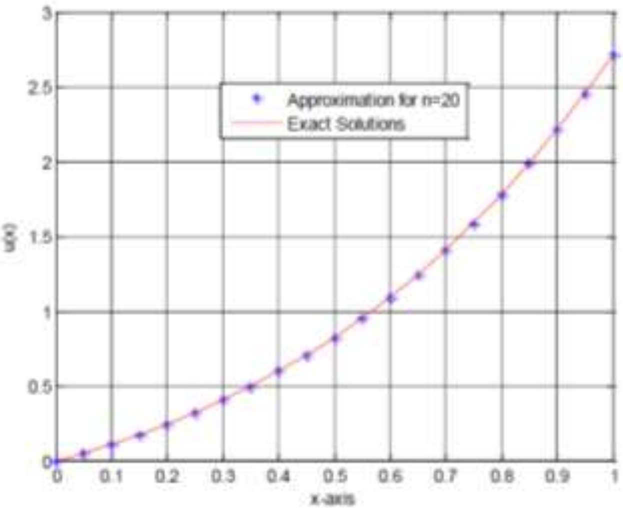


Figure (1): The exact and numerical solution of applying Algorithm (4.1) for equation(8).

The CPU time is 0.018776 seconds. Figure (2) shows the absolute error resulting of applying algorithm (4.1) for equation (8).

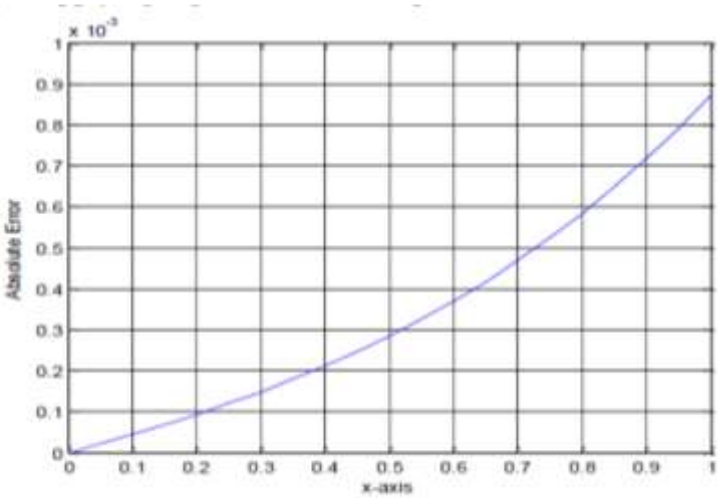


Figure (2) the error resulting of applying algorithm (4.1) on equation (8)

ii. NonlinerFredholm Integral Equations:

We try applying some of the numerical methods to approximation the solution of the FredholmInteggral Equation.

$$f(x)=\frac{2}{\pi}\cos(x)+\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\cos(x-y)\ f(y)dy \qquad (9)$$

This method include: the degenerate Kernel method, the collocation method and the Nusröm method, we will use suitable algorithums and Matlab software, then we will compare the exact solution with the approximate one using suitable number of n points

Note: the exact solution $f(x)=\sin x$ of the above integral equation (9) .

iii: TheNumericalRealizationofEquation(9)usingtheNyströmMethod:

TosolvetheFredholmintegralequationofthesecondkindwhichisgivenby

$$f(x)=-\frac{2}{\pi}\cos(x)+\frac{4}{\pi}\int_{0}^{\frac{\pi}{2}}\cos(x-y)\,f(y)dy$$

By Nyström method first we should remember that the kernel $\cos(x-y)$and the function $-\frac{2}{\pi}\cos(x)$ must continuous , secondly ,

We should know that we can approximate the integral $\int_a^b \emptyset\,(y)\,dy$ using quadrature rule by $\sum_{j=0}^{n} w_j\ \emptyset(y_j)$ by such approximation for $a\le x\le b$ the fredholm integral equation .

$$f(x)=g(x)=\lambda\int_D G(x,y)\,f(y)dy, \quad x\in \qquad (10)$$

canbe reducedto

$$f_n(x)=\lambda\sum_{j=1}^{n} w_i\,G(x,x_j)f_n(x_j)+g(x) \qquad (11)$$

where its solution $f_n(x)$ is an approximation of the exact solution $f(x)$ to(4.29). A solution to a functional equation (4.30) can be obtained if weassign x_i'stoxinwhichi=1,2,…,$n$and$a\le x_i\le b$.Inthisway,(11)is reducedtoa systemofequations

$$f_n(x_i)=\lambda\sum_{j=1}^{n} w_i\,G(x_i,x_j)f_n(x_j)+g(x_i) \qquad (12)$$

Next,writingtheequation(4.31)in thematrix form

$$F=\lambda kDF+G\rightarrow F-\lambda KDF=G\rightarrow(1-\lambda kD)F=G \qquad (13)$$

Where

$F=[f_n(x_i)]^T$, $G=[g(x_i)]^T$, $k=[G(x_i,x_j)]$,

$$D=diag(w_1,w_2,\dots,w_n)$$

$\int_a^b G(x\ \ ,y)=\sum w_j\,G(x_i,x_j)=Dk$ (14)

It'sworthtomentionthatinordertoapproximatetheintegral,wewillusetheTrapezoidal Rule.

Here,weimplement it in theformsuch that

$$\int_a^b G(x\ \ ,y)=\sum w_j\,G(x_i,x_j)=Dk \qquad (15)$$

where$D$isadiagonalmatrixsuchthattheelementsofitsdiagonalequal***h***

where***h***dependsontheinitialandtheendpointsoftheinterval$[a,b]$,andthenumberoftheapproximations***n***suchthat $h=\frac{b-a}{n}$.Theelementsof

thematrix$K$consistoftheentries$k(x_i,x_j)$,wherei,j=1,2,…,n,suchthattheapproximations$x's$obtainedas$x_i=a+h*i$,where i=2,3,…,n,

and$x_1=a$.

ThefollowingalgorithmimplementstheNyströmmethodusingtheMatlab software.

Algorithm (4.2):

In put $a, b, n, \lambda, g(x), G(x)$

$$h \to \frac{b-a}{n}$$

$$x_1 = a, x_n = b$$

for$i = 2\ to\ n-1$

$$x_i = a + h * i$$

end

for$i = 1\ to\ n$

$$G_i = g(x_i)$$

$$s_i = x_i$$

$D_{i1} = h \to D$is diagonal matrix

for$j = 1\ to\ n$

$$k_{ij} = k(x_i, x_j)$$

end

end

$$1 \to identity\ matrix$$

$$lhs \to 1 - \lambda Dk$$

$$F = lhs\ \ answer\ of\ lhs\ \ * f = G$$

$$p(f) \to the\ interpolating\ polynomial\ of\ \ [s_{i,} f_i]$$

Table (2) shows the exact solution $f(x)$ = sin(x) and the approximate onewhen n = 50 ,andshowing theerrorresultingofusingthenumericalsolution.

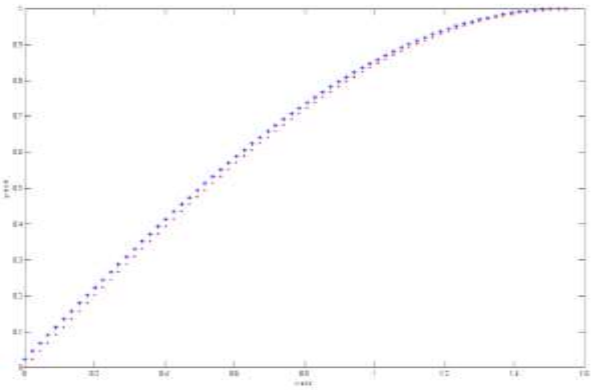
Note:Thetableshows thefirst10 values andthelast10 valuesonly

Table(2):TheExactandNumericalSolutionofapplyingAlgorithm(4.2)forequation (9).

x	Analyticalsolution y_1=sin(x)	Approximatesolution y_2	Error=$\|y_1 - y_2\|$
0	0	0.031405592470328	0.031405592470328
0.0314	0.031410759078128	0.062780191412531	0.031369432334402
0.0628	0.062790519529313	0.094092833885359	0.031302314356046
0.0942	0.094108313318514	0.125312618091103	0.031204304772588
0.1257	0.125333233564304	0.156408733871965	0.031075500307661
0.1571	0.156434465040231	0.187350493115954	0.030916028075723
0.1885	0.187381314585725	0.218107360042338	0.030726045456613
0.2199	0.218143241396543	0.248648981336784	0.030505739940241
0.2513	0.248689887164855	0.278945216106394	0.030255328941540
0.2827	0.278991106039229	0.308966165625180	0.029975059585951
1.2881	0.960293685676943	0.968423843447016	0.008130157770073
1.3195	0.968583161128631	0.975756237987680	0.007173076859049
1.3509	0.975916761938747	0.982125678925927	0.006208916987179
1.3823	0.982287250728689	0.987525880392547	0.005238629663858
1.4137	0.987688340595138	0.991951513040665	0.004263172445527
1.4451	0.992114701314478	0.995398209305166	0.003283507990688
1.4765	0.995561964603080	0.997862567712965	0.002300603109885
1.5080	0.998026728428272	0.999342156239842	0.001315427811571

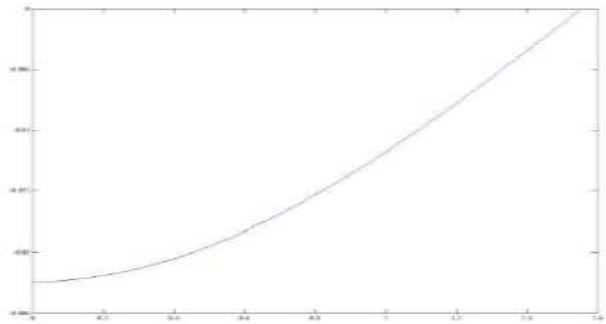
1.5394	0.999506560365732	0.999835514710546	0.000328954344814

Figure (3) compare the exact solution $f(x) = \sin x$with the approximate one when $n = 50$, while Figure 1.3 shows the error resulting of applying Algorithm (4.2) onthe equation (9), and how it approaches zero.



Figure(4.3):Theexactandnumericalsolutionof applyingAlgorithm(4.2)forequation(9).

TheCPU time is0.064010seconds.



Figure(4) :Theresultingerror ofapplyingalgorithm(4.2)to(9)

V. Result

We found that solution of nonlinear integral equations numerically using Matlab give us some important graphically solutions .These solutions moved away from the purely theoretical aspect and provided practical examples without prejudice to the scientific accuracy so that the information would be easy. There is a difference in the nonlinear Fredholm integral equation of the second kind and the homogeneous nonlinear Fredholm integral equation of the second kind and it was found that the nonlinear Fredholm integral equation of the second type has many ways to solve it. As for the integral nonlinear equations, there is only one way to solve it.Solving Nonlinear Integral Equations Numerically is very easy by usingMatlab.

VI. Conclusion

Wehaveusedthefollowingnumericalmethods: Trapezoidal Rule.for approximatingthesolutionofthe Volterra integralequations and Nyström methods, for approximatingthesolutionofthe Fredholmintegralequations.We have presented each numerical method as algorithm and appliedthese algorithms on the sameVolterra andFreedholm integral equation using MatlabSoftware; we have found that the numerical solution was approximately astheexactsolution.Theabsoluteerrorhasapproachedzerowhichwasshownthatnumericalresultswere acceptable.

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