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Calculating the Inverse Square Law over the Area of a Cardioid Surface

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ABSTRACT: *It is a fact in acoustics that doubling the distance from a sound source results in a -6dB loss. This is known as the Inverse Square Law (ISL). Generally, the justification of this law is limited to a geometric projection and proportion of concentric sphere surfaces. In the present paper the authors want to use an ideal loudspeaker with a cardioid radiation pattern to validate the ISL using two concentric cardioids one of which is twice the side of the other. In addition, the areas of these cardioids were calculated, and their ratios of radiation were compared.*

I. INTRODUCTION

It is well-known in acoustics that doubling the distance from its point of source the sound intensity results in a -6dB loss according to the Inverse Distance Law. Sound Intensity Level (SIL) units are measured in Acoustic Power by square meter (W/m^2)[1], however, for loudspeaker specifications it is preferred to use Decibel because rounded values are easier to manipulate. It is also known that the decibel of sound intensity (dB_{sil}) is the ratio between two intensity levels one of which is used as a reference value. In this paper the authors, to validate the ISL, use an arbitrary dB value coming from a sound source with a cardioid pattern assuming a uniform sound intensity over its surface. When considering the arbitrary dB value for the first cardioid, the smaller of the two, the authors convert the decibels to watts/m2 and divide this result by the area of its surface. Dividing this same quantity by the area of the second cardioid, the larger of two, and converting it to decibel results in a -6dB loss. As it is known, doubling the surface area of one cardioid with respect to another also doubles the linear distance between the cusp and the vertex [2]. Based on this latter fact the authors validate the ISL one of which fundamentals is the doubling the distance from the source.

The Sound Intensity Level Formula expressed in dB is $L=10*\log*(I_1/I_2)$ where I_1 is the Intensity of the source and I_2 the Sound Intensity of reference, to wit, of 10^{-12} W/m^2. [1]

An example of the Inverse Square Law using a sphere can be obtained by doubling its radius which, in turn, quadruples its surface area according to the formula, Surface of the sphere = $4\pi r^2$. Assuming a sound intensity I_1 (W/m2) and $I_2 = I_1/4$ (W/m2) then $I_1/I_2= 4$. This result shows that sound intensity of N_1 quadruples over the area N_2.

If we now compare the two quantities by the dB formula of the intensity of sound Li= 10*log*(I2/I1) we will get a total dB which has dropped[1] to Li=10*log(0.25) = -6.02.

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Doubling the distance between the cusp and the vertex of the cardioid is equivalent to doubling the circle diameter which quadruples its area (see Figure) [2,3]. By analogy we will get the same proportion within a sphere and a 3D cardioid.

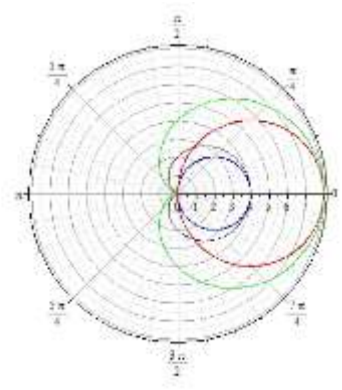

Figure 1 Maple software representation of circles and cardioids. plot({4*cos(x), 8*cos(x), 2+2*cos(x), 2*(2+2*cos(x))}, x = 0 .. 2*Pi, color = [blue, red, violet, green], coords = polar, axiscoordinates = polar)

II. AREA OF THE SURFACE OF A CARDIOID.

Now we will calculate the surface area of the cardioid

$r = a(1 + \cos\theta)$ (1)

We know that in the curve $r = r(\vartheta)$, r is the radial distance to the point P and θ is the angle of the ray to the point P with respect to the x-axis. See Figure 2.

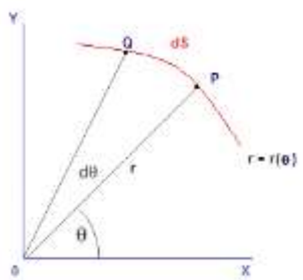

Figure 2 Element of arc length

The dS will be

$$dS = \sqrt{dx^2 + dy^2}$$

Converting to polar coordinates and knowing that
r = $r(\vartheta)$

$x = r * cos\theta$ $y = r * \sin\theta$

Then

$$dx^2 = (\cos(\theta) * dr + r(\theta) * (-sin\theta) * d\theta)^2$$

$$dy^2 = (\sin(\theta) * dr + r(\theta) * \cos(\theta) * d\theta)^2$$

$$dx^2 = \cos(\theta)^2 * dr^2 - 2 * dr * \cos(\theta) * r(\theta) * \sin(\theta)\, d\theta + r(\theta)^2 * \sin(\theta)^2 * d\theta^2$$

$$dy^2 = \sin(\theta)^2 * dr^2 + 2 * dr * \sin(\theta) * r(\theta) * \cos(\theta) d\theta + r(\theta)^2 * \cos(\theta)^2 * d\theta^2$$

Addition

$$\cos(\theta)^2 * dr^2 + r(\theta)^2 * Sin(\theta)^2 * d\theta^2 + \sin(\theta)^2 * dr^2 + r(\theta)^2 * \cos(\theta)^2 * d\theta^2$$

reordering, grouping and simplifying [4]

$$dS = \sqrt{dr^2(\sin(\theta)^2 + \cos(\theta)^2) + r(\theta)^2 d\theta^2(\sin(\theta)^2 + \cos(\theta)^2)}$$

$$dS = \sqrt{dr^2 + r(\theta)^2 d\theta^2}$$

factorizing and rearranging the terms we have that

$$ds = \sqrt{d\theta^2\left(r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right)}$$

$$ds = d\theta\sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Using the theorem for a polar equation (1) [5]

$$\frac{dS}{d\theta} = \sqrt{r^2 + (\frac{dr}{d\theta})^2}$$

$$\frac{dS}{d\theta} = \sqrt{a^2(1 + \cos(\theta))^2 + a^2 Sin(\theta)^2}$$

$$\frac{dS}{d\theta} = a\sqrt{1 + 2\cos(\theta) + \cos(\theta)^2 + \sin(\theta)^2}$$

$$\frac{dS}{d\theta} = a\sqrt{2 + 2\cos(\theta)}$$

Using the identity of the half-angle [5]

$$2\,cos^{2}\left(\frac{\theta}{2}\right) = 1 + \cos(\theta)$$

We get

$$\frac{dS}{d\theta} = a\sqrt{2(2cos^{2}\left(\frac{\theta}{2}\right))}$$

$$\frac{dS}{d\theta} = 2a\cos(\frac{\theta}{2})$$

Now, we can procced to calculate the surface of the revolving cardioid.

The integral is

$S = 2\pi \int_0^{\pi} y \frac{dS}{d\theta} d\theta$

$$S = 2\pi \int_0^{\pi} r * \sin(\theta) * 2aCos\left(\frac{\theta}{2}\right) d\theta$$

$$S = 4\pi a \int_0^{\pi} a(1 + \cos(\theta)) * \sin(\theta) * \cos\left(\frac{\theta}{2}\right) d\theta$$

Using the identity of

$$\sin\theta = 2\sin\frac{\theta}{2} * \cos\frac{\theta}{2}$$

Then we get

$$4\pi a^{2} \int_0^{\pi} 2cos^{2}\left(\frac{\theta}{2}\right) * 2\sin\left(\frac{\theta}{2}\right) * \cos\left(\frac{\theta}{2}\right) * \cos\left(\frac{\theta}{2}\right) d\theta$$

$$16\pi a^{2} \int_0^{\pi} cos^{4}\left(\frac{\theta}{2}\right) * \sin\left(\frac{\theta}{2}\right) d\theta$$

$$-32\pi a^{2} * \frac{Cos^{5}\frac{\pi^{5}}{2}}{5} |_0^{\pi}$$

$$S = -32\pi a^{2} * (0 - 1)$$

$$S = \frac{32\pi a^{2}}{5}$$

III. SOUND INTENSITY OVER FIRST CARDIOID

2.1 Acoustic Power of the Source

We will assume that our ideal cardioid loudspeaker has 94dB at a given distance. We must convert this decibel measure to acoustic power units (watts). To do this, we will use the dB Acoustic Power formula.

$$L_p = 10 * \log(\frac{P_1}{P_2})$$

Where L_p is acoustic power level.
P_1 is acoustic power de the loudspeaker measure
P_2 is the reference acoustic power (10^{-12}W)

$$P_1 = 10^{\frac{dB}{10}} * P_2$$

$$P_1 = 10^{\frac{94}{10}} * 10^{-12}$$

$$P_1 = 10^{9.4} * 10^{-12}$$

$$P_1 = 10^{-2.6} W$$

2.2 Area of the first cardioid.

Assuming that our first cardioid is $r = a(1+\cos\vartheta)$ with a = 1and using formula [1] we obtain the result shown in Figure 3.

$$S_1 = \frac{32\pi a^2}{5}$$

$$S_1 = \frac{32\pi}{5}$$

$$S_1 = 20.1$$

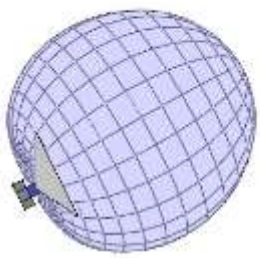

Figure 3 Loudspeaker into 3D cardioid

2.3 Sound Intensity

Knowing that the acoustic power level of the loudspeaker is $10^{-2.6}$ W, if we divide this power by the area of the surface of our first cardioid we obtain the sound intensity on the first cardioid

$$\frac{P_1}{S_1} = \frac{10^{-2.6}}{20.1}$$

$$\frac{P_1}{S_1} = 1.25 * 10^{-4}\frac{w}{m^2}$$

IV. SOUND INTENSITY OVER SECOND CARDIOID

3.1 Area of the second cardioid.

If our second cardioid is r = a(1+cosθ) with a = 2 and using formula [1], we obtain the result shown in Figure 4.

$$S = \frac{32\pi a^2}{5}$$

$$S = \frac{32\pi 2^2}{5}$$

$$S = 80.4$$

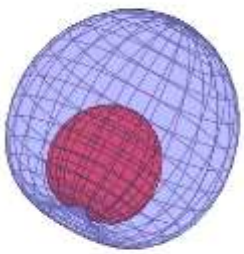

Figure 4 Concentric cardioids

3.2 Sound Intensity

Knowing that the sound intensity is 10^{-12}W by 1mt^2 then, there are 2*10^{-11}W/mt^2 on the 80.1 area of the cardioid.

The total acoustic power through the second cardioid is

$$\frac{P_1}{S_1} = \frac{10^{-2.6}}{80.1}$$

$$\frac{P_1}{S_1} = 3.1 * 10^{-5}$$

V. Reducing 6 dB by the Inverse Square Law

If we apply the decibel formula to the sound intensity we obtain

$$L_i = 10 * \log\left(\frac{3.1 * 10^{-5}}{1.25 * 10^{-4}}\right)$$

$$Li = 10 * -0.61dB$$

$$L_i = -6.1dB$$

Because the sound intensity is the square of the sound pressure level (SPL) if we now use the formula of the dBspl we obtain

$$L_{spl} = 20 * log\left(\frac{\sqrt{3.1 * 10^{-5}}}{\sqrt{1.25 * 10^{-4}}}\right)$$

$$L_{spl} = 20 * log\left(\frac{3.1 * 10^{-5}}{1.25 * 10^{-4}}\right)^{\frac{1}{2}}$$

$$L_{spl} = -6.1dB$$

VI. Conclusion

The Inverse Square Law is widely useful in science because several physical properties decrease their intensity according to this law. In this work, the authors have validated the ISL over the area of a cardioid radiation pattern for an ideal loudspeaker. One of the motivations for this work is that the area of the surface of a 3D cardioid is hardly ever considered in calculus textbooks. It has been the intention of the authors to fill this gap. In addition, the authors have verified that doubling the distance from a sound source results in a -6dB loss.

VII. REFERENCES

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THE AUTHORS

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José Mujica holds an Electrical Engineering degree from the Universidad Central de Venezuela. He is also an AES Fellow and author of several books including Audio Engineering (1990-2006), Audio Dictionary (1990 in Spanish), Infinitesimal Calculus with Analytic Geometry and Computing Applied to Audio (1993 in Spanish), and The Master handbook of P.A. (1993 In Spanish). He has written software for the electroacoustic Hewlett-Packard HP-41CV calculator (1984) using Constant Voltage 70.7V Software for Hewlett-Packard 86B (1988) and Q-Basic (1989). Mr. Mujica developed the software Audio Utilities with 20 applications (dB, RT60, Mass Law, 70.7V, filters, electroacoustic, etc.) in Visual Basic (1992). He has worked too as an Electroacoustic Consultant dedicated to acoustic prediction software such as EASE and CATT. In addition, Mr. Mujica is co-founder and Director of the School of Audio and Acoustics in Caracas, Venezuela. He is also author of several papers including Discrete Fourier Transform: two examples steps by step. Since 2015 he has been working on Audio Engineering Historical interviews and the Acoustic Prediction Channel on YouTube.

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