American Journal of Sciences and Engineering Research

E-ISSN -2348 – 703X, Volume 5, Issue 6, 2022

The Relationship between Surface, Linear and Volumetric Integral

Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman[1], Tasabeh Awadallah Abdel Majid Ali[2] and Guwyria Elgiely Eltahir Fadlamoola[3]

[1,2,3]*Department of Mathematics, Faculty of Eduction , Omdurman IsIamic University ,Omdurman, Sudan*

Abstract:*Surface integration is a double integration of linear integration. Closed surface integration can be linked with volumetric integration according to the divergence theorem, as well as between surface integration and linear integration according to Stokes theorem.*

Key Wards*: Calculation, Surface, Volumetric, Linear, Integral.*

I. Introduction

Integration is the reverse process of differentiation.Despite the multiplicity of definitions used for integration and the multiplicity of ways to use it, and that the result of these methods are all the same, all definitions ultimately lead to the same meaning.Integrals appear in many applied cases. If we consider a swimming pool, for example, if it is rectangular in length, width and depth, then it is possible to find the volume of water that can be contained (to fill it), its surface area (covered on all sides) and the length of its edges (with a rope, for example).But if they are oval and rounded at the bottom,thenallofthesequantitiescallforintegration. Applied approximations may suffice in such simple examples, but geometric precision requires exact values fortheseelements.Integrationisabranchofmathematics that deals with rates of change and motion, which is the opposite of differentiation, and this science arose from the desire to understand various physical phenomena such as the orbits of the planets and the effects of gravity. Calculus' success in formulating physical laws and predicting their results led to the development of a new sectionofmathematicscalledanalysis,ofwhichcalculusisa large part .Calculus is today the primary language of science and engineering, the primary means by which physical laws are expressed in mathematical terms, and an invaluable scientific tool in the further analysis of physical laws, in predictingthebehaviorofelectricalandmechanicalsystems governed by these laws and in discovering new laws And theprimaryuseoftheintegralisasacontinuousversionoftheprocessofaddition,butintegralsareoftencomputedby viewing the integral as essentially the opposite of differentiation. We will study about three types of integrals (surface, linear, volumetric) and we will know the theorems that connect them all.

II. Surface Integral

Definition (2.1):

A surface integral is a generalization ofmultipleintegralstointegrationoversurfaces.Itcanbethoughtof as the doubleintegralanalogueofthelineintegral.Givenasurface,onemayintegrate ascalarfield (that is, a function of position which returns a scalar as a value) over the surface, or a vector field(thatis, afunctionwhichreturnsa vector as value).If a region$R$ is not flat, then it is called a *surface* . [2]

Theareaofasurface$S$inR^3definedparametricallyby$r(u,v) =< x(u,v),y(u,v),z(u,v) >$

Overa regionofintegration$R$intheinput-variableplaneisgivenby

$$\int\int_S ds = \iint_R |r_u \times r_v| dA$$

Now,let$w = f(x,y,z)$beafunctiondefinedoverthissurface. Wethenwishtocalculate the surface integral, where $f\big(r(u,v)\big) = f\big(x(u,v),y(u,v),z(u,v)\big)$:

$$\iint_S f(r(u,v))\, ds = \iint_R f(r(u,v))|r_u \times r_v|\, dA$$

WhenthesurfaceSisdefinedexplicitlybyafunction$z = g(x,y)$,then$r(x,y) =$ $\langle x, y, g(x,y)\rangle$andthesurfaceintegralcanberewritten

$$\iint_S f(x,y,z)\, ds = \iint_R f(x,y,g(x,y)) \sqrt{g_x(x,y)^2 + (g_y(x,y))^2 + 1} dA$$

Where$dS = |r_u \times r_v| dA = \sqrt{g_x(x,y)^2 + (g_y(x,y))^2 + 1} dA$

Surfaceareaintegralsareaspecialcaseofsurfaceintegrals,where $f(x,y,z) = 1$. Surface integrals can be interpreted in many ways.

***Vector Areas of Surfaces*(2.2):**

VectorareasofsurfacesThevectorareaofasurfaceSisdefinedas

$$S = \iint_S dS$$

Example(2.3):Findthevectorareaofthesurfaceofthehemisphere $x^2 + y^2 + z^2 = a^2$with.$z \geq 0$

Solution:

$$dS = a^2 \sin\theta\, \hat{r}\, d\theta\, d\emptyset$$

in spherical polar coordinates .The vectorareais

$$S = \iint_S a^2 \sin\theta\, \hat{r}\, d\theta\, d\emptyset$$

Since$\hat{r}$variesoverthesurface$S$,italsomust beintegrated.OnSwehave

$$\hat{r} = \sin\theta\cos\varphi\, i + \sin\theta\sin\varphi\, j + \cos\theta\, k$$

So

$$S = i\left(a^2 \int_0^{2\pi} \cos\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta\right) + j\left(a^2 \int_0^{2\pi} \sin\theta\, d\varphi \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta\right) + k\left(a^2 \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta\right)$$

$$= 0 + 0 + \pi a^2 k = \pi a^2 k$$

Projectedareaofthehemisphereontothexy-plane.

III. Line Integral

Inmathematics,alineintegrals anintegralwherethe function to be integrated is evaluated along a curve.[6]

The terms path integral, curveintegral,andcurvilinearintegralarealsousedcontourintegralisusedaswell, although that is typically reserved for line integrals in the complex plane.[5]

The function to be integrated may be a scalar field or a vector field. The .value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve (commonly arc length or, for a vector field, the scalar product of the vector field with a differential vector in thecurve).Thisweightingdistinguishesthelineintegralfromsimplerintegrals defined on intervals.

Line Integral of Scalar Functions(3.1):

If f is a continuous function whose domain includes a smooth (or piecewise smooth) curve Cin R^n , we can integrate f over the curvetaking the differential in the integral to be the element of arc length dS.thus , if $C$is parametrized by $x = g(t), a \leq t \leq b$, we define

$$\int_C fdS = \int_a^b f(g(t))|\acute{g}(t)|dt$$

This is independent of the parametrization and the orientation by the same chain rule calculation that we performed above for the case $f \equiv 1$.

Line Integral of Vector Field:(3.2)

We can define the integral of an R^m valued function over a curve in ,R-$n$simply by integrating each component separately that is F=(F-1.,F-2.,…,,F-m.),then ,C- -FdS.=(,C- -,F-1.dS.,..,C- -,F-m.dS.) There is not much to be said about the such integrals that does not follow immediately from the fact scalar valued integrals.

Proposition(3.3):

If F is continuous R^m valued function on a, then

a-b-F(t).dt≤,,b-a-F(t)..dt [1]

Complex line integral (3.4):

In complex analysis, the line integral is defined in terms of multiplicationandadditionofcomplex numbers.Suppose$U$isan opensubsetofthecomplex plane$C$, $f: U \to C$isa function and $L \subset U$is curveoffinitelength,parameterizedby$\boldsymbol{Y}: [\boldsymbol{a}, \boldsymbol{b}] \to \boldsymbol{L}$ where$Y(t) = x(t) + iy(t)$Thelineintegral

$$\int_l f(z)dz$$

Maybedefinedbysubdividingtheinterval$[a, b]$into

$$a = t_0 < t_1 < \cdots < t_n = b$$

Andconsideringtheexpression

$$\sum_{k=1}^{n} \boldsymbol{f}(\boldsymbol{Y}(\boldsymbol{t_k}))\,[\boldsymbol{Y}(\boldsymbol{t_k}) - \boldsymbol{Y}(\boldsymbol{t_k} - \mathbf{1})] = \sum_{k=1}^{n} \boldsymbol{f}(\boldsymbol{Yk})\,\Delta\boldsymbol{Yk}$$

TheintegralisthenthelimitofthisRiemannsumasthelengthsof the subdivision intervals approach zero. Iftheparameterizationiscontinuouslydifferentiable,theline integral can be evaluated as an integral of a function of a real variable: [5]

$$\int_L \boldsymbol{f}(\boldsymbol{z})\,\boldsymbol{dz} = \int_a^b \boldsymbol{f}(\boldsymbol{Y}(\boldsymbol{t}))\boldsymbol{Y}(\boldsymbol{t})\,\boldsymbol{dt}$$

When$\boldsymbol{L}$ isaclosedcurve(initialandfinalpointscoincide),theline

integralisoftendenoted$\oint_L \boldsymbol{f}(\boldsymbol{z})\boldsymbol{dz}$

sometimesreferredtoinengineeringgas acyclicintegral.

Thelineintegralwithrespecttotheconjugatecomplex differential$\overline{\boldsymbol{dz}}$ defined to be

$$\int_L \boldsymbol{f}(\boldsymbol{z})\overline{\boldsymbol{dz}} = \overline{\int_L \overline{\boldsymbol{f}(\boldsymbol{z})}\,\boldsymbol{dz}} = \int_a^b \boldsymbol{f}(\boldsymbol{Y}(\boldsymbol{t}))\,\overline{\boldsymbol{Y}(\boldsymbol{t})}\boldsymbol{dt}$$

The line integrals of complex functions can be evaluated using a number of techniques. The most direct is to split into real and imaginaryparts,reducingtheproblemtoevaluatingtworeal-valued line integrals. The Cauchy integral theorem may be used to equate the line integral of an analytic function to the same integral over a more convenient curve. It also implies that over a closed curve enclosing a region where$\boldsymbol{f}(\boldsymbol{z})$ is analytic without singularities, the value of the integral is simply zero, or in case the region includes singularities,theresiduetheoremcomputestheintegralintermsof the singularities.

Example(3.5)

Considerthefunction$\boldsymbol{f(z)} = \frac{1}{z}$andletthe contour$\boldsymbol{L}$bethecounterclockwiseunitcircleabout$\mathbf{0}$,parametrizedby$\boldsymbol{z(t)} = \boldsymbol{e^{it}}$ with $\boldsymbol{t}$ in $[\mathbf{0}, \mathbf{2\pi}]$ using the complex exponential. Substituting, we find:

$$\oint \frac{1}{z}_L dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = i\int_0^{2\pi} e^{-it} e^{it} dt = i\int_0^{\pi} dt = i(2\pi - 0) = 2\pi i$$

Relation of Complex Line Integral and Line Integral of Vector Field(3.6):

Viewingcomplexnumbersas2-dimensionalvectors,thelineintegral of a complex-valuedfunction$\boldsymbol{f(z)}$has realand complex parts equal to the line integral and the flux integralof the vector fieldcorrespondingtotheconjugatefunction$\overline{f(z)}$Specifically,if
$r(t) = (x(t), y(t))$parametrizes$L$,and

$$\boldsymbol{f(z) = u(z) + iv(z)\}f(z) = u(z) + iv(z)}$$

corresponds to the vector field$F(x, y) = \overline{f(x + iy)} = (u(x + iy), -v(x + iy))$then:

$$\int_L f(z)dz = \int_L (u + iv)(dx + idy) = \int_L (u, -v) \cdot (dx, dy) + \int_L (u, -v) \cdot (dy, dx)$$

$$\int_L F(r) \cdot dr + i\int_L F(r) \cdot dr^{\perp}$$

By Cauchy's theorem, the left-hand integral is zero when$\boldsymbol{f(z)}$is analytic (satisfying the Cauchy–Riemann equations) for any smooth closedcurve$L$Correspondingly,byGreen'stheorem,theright-hand integrals are zero when $F = \overline{f(z)}$is irrotational (curl-free) and incompressible(divergence-free).Infact,theCauchy-Riemann equations for$f(z)$are identical to the vanishing of curl and divergence for F.

IV. Volume Integral

In mathematics (particularly multivariable calculus),a volume integral refers to an integral over a 3-dimensionaldomain;that is, it is a specialcaseof multiple integrals.Itcanalsomeanatripleintegralwithinaregion $\boldsymbol{D} \subset \boldsymbol{R}^3$afunction$\boldsymbol{f(x, y, z)}$,andis usuallywritten as**[6]:**

$$\iiint_D f(x, y, z)dxdydz$$

Avolumeintegralincylindricalcoordinatesis

$$\iiint_D f(\rho, \varphi, z)d\rho d\varphi dz$$

Andavolumeintegralinsphericalcoordinates(usingthe ISO convention for angles with$\boldsymbol{\varphi}$ as the azimuth and θmeasuredfromthepolaraxis(seemoreonconventions)) has the form

$$\iiint_D f(r, \theta, \varphi)r^2 \sin\theta \, dr d\theta d\varphi$$

Volumeintegrals:

$$\int_V \varphi dV \quad , \int_V a dV$$

Volumes of Three-Dimensional Regions (4.1):

Thevolumeofathree-dimensional region$V$issimplpy$V = \int_V \ dV$.

Weshallnoweexpressitintermsofasurfaceintegralover$\boldsymbol{S}$.

Ageneralvolume$\boldsymbol{V}$containingtheoriginandboundedbytheclosed surface $\boldsymbol{S}$.Letussupposethattheorigin$\boldsymbol{O}$is containedwithinthe$\boldsymbol{V}$.ThenThe volumeofthesmallshadedconeis $\boldsymbol{dV} = \frac{1}{3}\boldsymbol{r} \cdot \boldsymbol{ds}$

Thetotalvolumeoftheregion isthengivenby

$$\boldsymbol{V} = \frac{1}{3}\oint_S \ \boldsymbol{r} \cdot \boldsymbol{ds}$$

Thisexpressionisstillvalidevenwhen$\boldsymbol{O}$ isnotcontainedin$\boldsymbol{V}$.

e volume enclosed between a sphere of radius a centeredontheorigin,andacircularconeofhalfangleαwithits vertex at the origin.[6]

Solution:

Now $dS = a^2 \sin\theta \, d\theta d\emptyset \hat{r}$ Taking the axis of the cone to lie along thez-axis(fromwhichθismeasured)therequiredvolumeisgivenby

$$\boldsymbol{V} = \frac{1}{3}\oint_S \ \boldsymbol{r} \cdot \boldsymbol{dS} = \frac{1}{3}\int_0^a \boldsymbol{d\emptyset} \int_0^a \boldsymbol{a^2 \sin\theta \, r} \cdot \hat{\boldsymbol{r}} \boldsymbol{d\theta}$$

$$\frac{1}{3}\int_0^{2\pi} \boldsymbol{d\emptyset} \int_0^a \boldsymbol{a^3 \sin\theta \, d\theta} = \frac{2\pi \boldsymbol{a}^3}{3}(1 - \cos \boldsymbol{a})$$

Integralformsforgrad,divandcurlAtanypoint$\boldsymbol{P}$,wehave

$$\nabla\emptyset = \lim_{V\to 0}\left(\frac{1}{V}\oint_S \ \emptyset \boldsymbol{dS}\right) \qquad (4.1)$$

$$\nabla \cdot \boldsymbol{a} = \lim_{V\to 0}\left(\frac{1}{V}\oint_S \ \boldsymbol{dS} \times \boldsymbol{a}\right) \qquad (4.2)$$

$$(\nabla \times \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = \lim_{A\to 0}\left(\frac{1}{A}\oint_S \ \boldsymbol{da} \cdot \boldsymbol{dr}\right) \qquad (4.3)$$

where$\mathbf{V}$isasmallvolumeenclosing$\mathbf{P}$and$\mathbf{S}$isitsboundingsurface.$\mathbf{C}$is a plane contour area $\mathbf{A}$ enclosing the point $\mathbf{P}$ and $\hat{\boldsymbol{n}}$ is the unit normal to the enclosed planar area.

Divergence Theorem and Related Theorems(4.2):

Imagine a volume $\mathbf{V}$,in which a vector field a is continuous anddifferentiabletobedividedupintoalarge number of small volumes $\boldsymbol{V_i}$Using Eq$(\mathbf{4.2})$ we have for each small volume

$$(\nabla \cdot \mathbf{a})\mathbf{V_i} \approx \oint_{\mathbf{S_i}} \ \mathbf{a} \cdot \mathbf{dS}$$

where$\boldsymbol{S_i}$ is the surface of the small volume$\boldsymbol{V_i}$.Summing over $\boldsymbol{i}$, contributionsfromsurfaceelementsinteriorto$\boldsymbol{S}$cancel,sinceeach surface element appears in two terms with opposite signs. Only contributions from surface elements which are also parts of $\boldsymbol{S}$survive. If each $\boldsymbol{V_i}$ is allowed to tend to zero, we obtain the divergence theorem

$\int_V \ \boldsymbol{\nabla} \cdot \boldsymbol{adV} = \oint_S \ \boldsymbol{a} \cdot \boldsymbol{dS}$ *(E)*

Ifwe set$\boldsymbol{a} = \boldsymbol{r}$*,we obtain*

$$\int_V \ \boldsymbol{\nabla} \cdot \boldsymbol{rdV} = \int_V \ \boldsymbol{3dV} = \boldsymbol{3V} = \oint_S \ \boldsymbol{r} \cdot \boldsymbol{dS}$$

Example (4.3):

Evaluatethesurfaceintegral $I = \int_S a \cdot dS$where

$a = (y - x)i + x^2zj + (z + x^2)k$,and$S$ is theopensurfaceof the hemisphere

$$x^2 + y^2 + z^2 = a^2, z \geq 0.$$

Solution :

Let us consider a closed surface $S' = S + S_1$, where$S_1$is the circular area in thexyplanegiven by $x^2 + y^2 \leq a^2$, zthenenclosesa hemisphericalvolume V. By the divergence theorem, we have$\int_V \nabla \cdot a dV = \oint_{S'} a \cdot dS = \int_S a \cdot dS + \int_{S_1} a \cdot dS$

Now$\nabla \cdot a = -1 + 0 + 1 = 0$,sowecanwrite$\int_S adS = -\int_S adS$

Thesurfaceelement on$S_1$is$dS = -kdxdy$.

On S_1we alsohave$a = (y - x)i + x^2k$,sothat

$$I = -\int_{S_1} a \cdot dS = \iint_R x^2\, dxdy$$

,where$R$ isthe circular region inthexy-planegivenby

$$x^2 + y^2 \leq a^2$$

Transformingtoplanepolar coordinates,we have

$$I = \iint_{R'} \rho^2 \cos^2 \emptyset \rho\, d\rho d\emptyset = \int_0^{2\pi} \cos^2 \emptyset\, d\emptyset \int_0^a \rho^2 d\rho = \frac{\pi a^4}{4}$$

V. Green's Theorem

Consider two scalar functions $\emptyset$andφthat are continuous and differentiableinsomevolume$V$ boundedbyasurface$S$.Applying the divergence theorem to the vector field$\emptyset\nabla\varphi$,we obtain (8)

$$\oint_S \emptyset\nabla\varphi \cdot dS = \int_V \nabla \cdot (\emptyset\nabla\varphi)dV = \int_V [\emptyset\nabla^2\varphi + (\nabla\emptyset) \cdot (\nabla\varphi)]dV \qquad (4.4)$$

Reversing therolesof $\emptyset$and φ in Eq.(4.4)andsubtracting thetwo equations gives

$$\oint_S (\emptyset\nabla\varphi - \varphi\nabla\emptyset) \cdot dS = \int_V (\emptyset\nabla^2\varphi - \varphi\nabla^2\emptyset)\, dV \qquad (4.5)$$

Equation(4.4)isusuallyknownasGreen'sfirsttheoremand(4.5)ashis second.

Other Related Integral Theorems (5.1):

If $\emptyset$is a scalar field and b is a vector field and both satisfy the differentiabilityconditionsinsomevolume$V$boundedbyaclosed surface S, then

$$\int_V \nabla\emptyset dV = \oint_S \emptyset dS \qquad (4.6)$$

$$\int_V \nabla \times b dV = \oint_S dS \times b \qquad (4.7)$$

Proofof Eq(4.6):

InEq.(E),let$a = \emptyset c$,where$c$isa constantvector.Wethenhave

$$\int_V \nabla \cdot (\emptyset c)dV = \oint_S \emptyset c \cdot dS$$

ExpandingouttheintegrandontheLHS,wehave

$$\nabla \cdot (\emptyset c) = \emptyset\nabla \cdot c + c \cdot \nabla\emptyset = c \cdot \nabla\emptyset$$

Also, $\emptyset c \cdot dS = c \cdot \emptyset dS$,sowe obtain

$$c \cdot \int_V \nabla\emptyset dV = c \cdot \oint_S \emptyset dS$$

Since$c$isarbitrary,weobtainthestatedresult.

Example(5.2): For a compressible fluid with time-varying position-dependentdensity$\boldsymbol{\rho(r,t)}$andvelocityfield$\boldsymbol{v(r,t)}$inwhichfluidis neither being created nor destroyed, show that

$$\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho v}) = \mathbf{0}$$

Solution:

For an arbitrary volume in the fluid, conservation of mass says that the rate of increase or decrease of the mass $\boldsymbol{M}$ in the volume must equalthenet rateatwhichfluidisenteringorleavingthe volume,i.e.

$$\frac{\boldsymbol{dM}}{\boldsymbol{dt}} = -\oint_{S} \boldsymbol{\rho v} \cdot \boldsymbol{dS}$$

where$\boldsymbol{S}$isthesurfacebounding$\boldsymbol{V}$. Butthemassoffluidin$\boldsymbol{V}$is

$\boldsymbol{M} = \int_{V} \boldsymbol{\rho dV}$, sowe ha

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\int_{V} \boldsymbol{\rho dV} + \oint_{S} \boldsymbol{\rho v} \cdot \boldsymbol{dS} = \mathbf{0}$$

Usingthedivergencetheorem,wehave

$$\int_{V} \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} \boldsymbol{dV} + \int_{V} \boldsymbol{\nabla} \cdot (\boldsymbol{\rho v})\boldsymbol{dV} = \int_{V} \left[\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho v})\right] \boldsymbol{dV} = \mathbf{0}$$

Sincethevolume$\boldsymbol{V}$isarbitrary,theintegrandmustbeidentically zero, so

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v)$$

VI. Stokes' Theorem and Related Theorems

Following the same lines as for the derivation of the divergence theorem, we can divide the surface S into many small areasS_i with boundaries $C_i$and with unit normal $\hat{n}_i$.Using Eq. (4.3) we have for each small area

$$(\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \hat{\boldsymbol{n}}_{\boldsymbol{i}} \approx \oint_{C_i} \boldsymbol{a} \cdot \boldsymbol{dr}$$

Summing over$\boldsymbol{i}$ we find that on the $\boldsymbol{RHS}$ all parts of all interior boundariesthatarenotpartof$\boldsymbol{C}$ areincludedtwice,beingtraversed in opposite directions on each occasion and thus contributing nothing. Only contributions fromline elements that are also parts of C survive. If each$\boldsymbol{S_i}$is allowed to tend to zero, we obtain Stokes' theorem

$$\int_{S} (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \boldsymbol{dS} = \oint_{C} \boldsymbol{a} \cdot \boldsymbol{dS} \; (\mathbf{6.1})$$

Example(6.1): Given thevector field$\boldsymbol{a = yi - xj + zk}$,verifyStokes' theoremfor the hemispherical surface$x^2 + y^2 + z^2 = a^2, z \geq 0$

Solution:

Letusevaluatethesurface integral

$$\int_{S} (\nabla \times a) \cdot dS$$

Overthehemisphere.Since$\boldsymbol{\nabla \times a = -2k}$ andthesurfaceelement is$dS = a^2 \sin\theta \, d\theta d\varphi \hat{r}$wehave

$$\int_{S} (\nabla \times a) \cdot dS = \int_{0}^{2\pi} d\varphi \int_{0}^{\frac{\pi}{2}} d\theta(-2a^2 \sin\theta)\hat{r} \cdot k = -2a^2 \int_{0}^{2\pi} d\varphi \int_{0}^{\frac{\pi}{2}} \sin\left(\frac{z}{a}\right) d\theta$$

$$= -2a^2 \int_{0}^{2} d\varphi \int_{0}^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta = -2\pi a^2$$

Thelineintegralaroundtheperimetercurve$C$(acircle$x^2 + y^2 = a^2$inthexy-plane)isgivenby

$$\oint_{C} a \cdot dr = \oint_{C} (yi - xj + zk) \cdot (dxi + dyj + dzk) = \oint_{C} (ydx - xdy)$$

Usingplanepolarcoordinates,on$C$wehave
$x = a\cos\varphi, y = a\sin\varphi$so that$dx = -a\sin\varphi\, d\varphi, dy = a\cos\varphi\, d\varphi$
and the line integral becomes

$$\oint_C (ydx - xdy) = -a^2\int_0^{2\pi}(\cos^2\varphi + \sin^2\varphi)d\varphi$$
$$= -a^2\int_0^{2\pi} d\varphi = -2\pi a^2$$

Sincethesurfaceandlineintegralshavethesamevalue,wehave verified Stokes' theorem.

VII. Related Integral Theorems

$$\int_S \boldsymbol{dS} \times \boldsymbol{\nabla\varphi} = \oint_C \boldsymbol{\varphi dr} \qquad (\mathbf{7.1})$$

$$\int_S (\boldsymbol{dS} \times \boldsymbol{\nabla}) \times \boldsymbol{b} = \oint_C \boldsymbol{dr} \times \boldsymbol{b} \qquad (\mathbf{7.2})$$

Proofof Eq.$(\mathbf{7.1})$
InStokes'theorem, Eq.$(\mathbf{6.1})$,let$\boldsymbol{a} = \boldsymbol{\varphi c}$,where$\boldsymbol{c}$isa constantvector.
We then have

$$\int_S [\boldsymbol{\nabla} \times (\boldsymbol{\varphi c})] \cdot \boldsymbol{dS} = \oint_C \boldsymbol{\varphi c} \cdot \boldsymbol{dr} \qquad (\mathbf{7.3})$$

Expandingouttheintegrand ontheLHS,wehave

$$\boldsymbol{\nabla} \times (\boldsymbol{\varphi c}) = \boldsymbol{\nabla\varphi} \times \boldsymbol{c} + \boldsymbol{\varphi\nabla} \times \boldsymbol{c} = \boldsymbol{\nabla\varphi} \times \boldsymbol{c}$$

Sothat

$$[\boldsymbol{\nabla} \times (\boldsymbol{\varphi c})] \cdot \boldsymbol{dS} = (\boldsymbol{\nabla\varphi} \times \boldsymbol{c}) \cdot \boldsymbol{dS} = \boldsymbol{c} \cdot (\boldsymbol{dS} \times \boldsymbol{\nabla\varphi}).$$

SubstitutingthisintoEq.(M),andtakingcoutofbothintegrals,we find

$$\boldsymbol{c} \cdot \int_S \boldsymbol{dS} \times \boldsymbol{\nabla\varphi} = \boldsymbol{c} \cdot \oint_C \boldsymbol{\varphi dr}$$

Sincecisaarbitraryconstantvector,wethereforeobtainthestated result.

Example(7.1): Integrating theequation$\boldsymbol{f(x,y,z)} = \mathbf{1}$overa unitcube yields the following result:

$$\int_0^1\int_0^1\int_0^1 \mathbf{1}\,\boldsymbol{dxdydz} = \int_0^1\int_0^1 (\mathbf{1}-\mathbf{0})\,\boldsymbol{dydz} = \int_0^1 (\mathbf{1}-\mathbf{0})\,\boldsymbol{dz} = \mathbf{1} - \mathbf{0} = \mathbf{1}$$

So thevolume of the unit cubeis 1 as expected. This is rather trivial however,andavolumeintegralisfar morepowerful.Forinstanceif we have a scalar density function on the unit cube then the volume integral will give the total mass of the cube. For example for density function:

$$\begin{cases} \boldsymbol{f}: \mathbb{R}^2 \to \mathbb{R} \\ \boldsymbol{f}: (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \leftrightarrow \boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z} \end{cases}$$

TheTotalMassoftheCube:
is$\int_0^1\int_0^1\int_0^1(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})\,\boldsymbol{dxdydz} = \int_0^1\int_0^1(\frac{1}{2}-\boldsymbol{y}+\boldsymbol{z})\,\boldsymbol{dydz} = \int_0^1(\mathbf{1}+\boldsymbol{z})\,\boldsymbol{dz} = \frac{3}{2}$ [9]

VIII. References

[1] Gerald B. Folland ,Advanced Calculus,Pearsion,217.
[2] Hazewinkel, Michiel (2001), Encyclopedia of Mathematics,Springer ,SurfaceIntegral.
[3] HELM(VERSION1:April14,2004):Workbook Level 1, 29,.2, Surface and volume Integrals
[4] JAMES STEWART , Calculus ,Eighth Edition, Cengage learning , USA,2015.
[5] Kwong-Tin Tang (30 November 2006), MathematicalMethodsforEngineersandScientists Vector Analysis, Ordinary Differential Equations and Laplace Transforms, Springer Science & Business Media, ISBN 978-3-540-30268-1.

[6] "Multipleintegral",EncyclopediaofMathematics, EMS Press, 2001, 1994.
[7] Nykamp,Duane,"Lineintegralsareindependentof parametrization" ,Math Insight. Retrieved September 18, 2020.
[8] Seymour Lipschutz,ph.,,Vector Analysis, Second Edition, MCGrawHill , New York,2009.
[9] Weisstein,EricW."Volumeintegral",MathWorld.
[10] Werner w. Rogosinski, Oliver and Boyd, London,1952.
[11] "ListofCalculusandAnalysisSymbols", Math Vault. 2020-05-11. Retrieved 2020-09-18.
[12] "16.2LineIntegrals".www.whitman.edu.Retrieved 2020-0 18.Ahlfors,Lars(1966).ComplexAnalysis(2nded.). New York: McGraw-Hill. p. 103.