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Numerical Resolution of the Unsteady Equations of Magnetohydrodynamic at Low Magnetic Reynolds Numbers Using Uzawa Method

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ABSTRACT: *Magnetohydrodynamic is essential to many applications in the domain of industry and engineering. This affects the heating or the control of the movements of conductive fluid. Therefore, solving the magnetohydrodynamic equations is essential to understand the phenomena generate by the coupling between hydrodynamics and electromagnetism. In this paper, we propose a numerical resolution of the equations of magnetohydrodynamic at low magnetic Reynolds number. Our approach use characteristic method, a fully coupled time discretization with Euler implicit scheme, lagrangian method and Uzawa algorithm. This approach allow to absorb the nonlinearity and to obtain stability and convergence. To assess the effectiveness of our approach, we compare the numerical solution with exact solutions in the unit cube. To ensure the similarity, we compute the space error and the rate of convergence on the velocity and on electric potential in the Lebesgue and Sobolev norm. The results of numerical experiments showed that our method are in good agreement with the analytical solutions and converge with a good accuracy.*

Key words: *Magnetohydrodynamic, characteristic method, finite element method, Uzawa algorithm, FreeFem++*

I. Introduction

In recent years, magnetohydrodynamic has aroused major interest due to its many applications in industry and engineering [8, 9]. Magnetohydrodynamic is branch of physics that studies the interaction between hydrodynamics and electromagnetism [9]. In fact, magnetohydrodynamic intervene whether it is to heat, set movement or control a fluid which conducts electric current (see [12] and [13]). We can cite for example the propulsion in sea water [10], the electrolysis of aluminum [11]. Thus, it is essential to solve the equations that govern magnetohydrodynamic to understand the behavior of electrically conductive fluids in the presence of an electromagnetic field. In this paper, we will consider the simplified magnetohydrodynamic equations for the case of low magnetic Reynolds numbers (we can see [6] and [7]) and which are valid for

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industrial application. In the literature, there are numerous method to solve numerically the time dependent simplified magnetohydrodynamic equations ([6], [7]) but most of them are constrained. The main difficulties in solving these equations are due to the nonlinearity and the coupling between Navier – Stokes equations and those of electromagnetism. To address this problem, we propose a characteristic method to treat the nonlinear term. In addition, we apply fully coupled time discretization and the lagrangian method. We will use Uzawa method for linear saddle point systems obtained. Our approach differs from existing methods in two important ways. First, our approach is suitable for numerical stability. The second, our method converge with a good accuracy.

II. Material and methods

1.1. Problem formulation

Let $\Omega \subset \mathbb{R}^d (d \leq 3)$ be a domain, representing a region of space. The domain Ω will always be assumed to be bounded and regular. The boundary of Ω is denoted by $\partial\Omega$ and is supposed to be at least Lipschitz continuous. The time – dependent simplified magnetohydrodynamic at low magnetic Reynolds numbers is modelled by: given time $T > 0$, body force $\boldsymbol{f}$, interaction parameter $N > 0$, Hartmann number M and $\boldsymbol{B}$ an external magnetic field, find velocity $\boldsymbol{u}: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, pressure $p: \Omega \times [0, T] \rightarrow \mathbb{R}$ and electric potential field $\phi: \Omega \times [0, T] \rightarrow \mathbb{R}$ satisfying:

$$\begin{cases} \dfrac{1}{N}\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u}\right) - \dfrac{1}{M^2}\boldsymbol{\nabla}^2\boldsymbol{u} + \boldsymbol{\nabla} p + \boldsymbol{\nabla}\phi \times \boldsymbol{B} - (\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{f} \\ \boldsymbol{\nabla}.\boldsymbol{u} = 0 \\ -\Delta\phi + \boldsymbol{\nabla}.(\boldsymbol{u} \times \boldsymbol{B}) = 0 \end{cases} \tag{2.1}$$

Subject to initial condition

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, \qquad \forall \boldsymbol{x} \in \Omega \tag{2.2}$$

and homogenous boundary condition

$$\boldsymbol{u}(\boldsymbol{x}, t) = 0, \qquad \phi(\boldsymbol{x}, t) = 0, \qquad \forall(\boldsymbol{x}, t) \in \partial\Omega \times [0, T] \tag{2.3}$$

1.2. Time discretization

In order, to absorb the non-linearity we introduce, $\forall \boldsymbol{x} \in \bar{\Omega}, \mathrm{t} \in [0, T]$ the characteristic, associated with the velocity field, define by $\boldsymbol{X} = \boldsymbol{X}(\tau; t, \boldsymbol{x})$ and which verify:

$$\begin{cases} \dfrac{d}{d\tau}\boldsymbol{X}(\tau; t, \boldsymbol{x}) = \boldsymbol{u}[\boldsymbol{X}(\tau; t, \boldsymbol{x}), \tau] \\ \boldsymbol{X}(t; t, \boldsymbol{x}) = \boldsymbol{x} \end{cases} \tag{2.4}$$

Thus, we can rewrite the equations (2.1) in the lagrangian form:

$$\begin{cases} \dfrac{1}{N}\dfrac{d}{d\tau}\left[\boldsymbol{u}[\boldsymbol{X}(\tau; t, \boldsymbol{x}), \tau]\right]_{\tau = t} - \dfrac{1}{M^2}\boldsymbol{\nabla}^2\boldsymbol{u} + \boldsymbol{\nabla} p + \boldsymbol{\nabla}\phi \times \boldsymbol{B} - (\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{f} \\ \boldsymbol{\nabla}.\boldsymbol{u} = \boldsymbol{0} \\ -\Delta\phi + \boldsymbol{\nabla}.(\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{0} \end{cases} \tag{2.5}$$

In this paper, we choose the semi – implicit Euler scheme. Let us denote $\Delta t > 0$ the time step with $T = n\Delta t$ and $(\boldsymbol{u}^{\boldsymbol{n}}, \phi^n, p^n)$ the approximate value at time t^n. The unknown fields at time t^{n+1} are obtained by solving:

$$\begin{cases} \dfrac{1}{N}\left(\dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n o \boldsymbol{X}^n}{\Delta t}\right) - \dfrac{1}{M^2}\boldsymbol{\nabla}^2\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} + \boldsymbol{\nabla} p^{n+1} + \boldsymbol{\nabla}\phi^{n+1} \times \boldsymbol{B} - (\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{f} \\ \boldsymbol{\nabla}.\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} = \boldsymbol{0} \\ -\Delta\phi^{n+1} + \boldsymbol{\nabla}.(\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}} \times \boldsymbol{B}) = \boldsymbol{0} \end{cases} \tag{2.6}$$

With boundary condition

$$\boldsymbol{u}^{n+1} = 0, \qquad \phi^{n+1} = 0, \qquad \forall x \in \partial\Omega \tag{2.7}$$

1.3. Weak formulation and saddle – point problem

The time semi – discretization is to solve a succession of continuous problems in shape space. Thus, we consider the problem formulated in Hilbert spaces, of infinite dimension. Suppose then that $L^2(\Omega)$ is the space of summbable square function on Ω, equipped with usual norm $\|.\|_{L^2(\Omega)}$and inner product $(.,.)$. We use the usual Sobolev spaces $H^k(\Omega)$ with his norm $\|.\|_k$ and $H_0^1(\Omega)$. We denote $H^{-k}(\Omega)$ the dual spaces of $H_0^1(\Omega)$. In addition, we define

$$\boldsymbol{X} = [H_0^1(\Omega)]^d = \left\{\boldsymbol{v} \in [H^1(\Omega)]^d \mid v_{|\partial\Omega} = 0\right\}$$
$$Y = H_0^1(\Omega) = \left\{\psi \in H^1(\Omega) \mid \psi_{|\partial\Omega} = 0\right\}$$
$$Q = L_0^2(\Omega) = \left\{q \in L^2(\Omega) \mid \int_\Omega q = 0\right\}$$

Therefore, we introduce the product spaces $\boldsymbol{W}$ as $\boldsymbol{W} = \boldsymbol{X} \times Y$ equipped with the norm $\|\tilde{\boldsymbol{v}}\|_1 = \|\boldsymbol{v}\|_1 + \|\psi\|_1$ for $\tilde{\boldsymbol{v}} = (\boldsymbol{v};\psi) \in \boldsymbol{W}$ and $V = \{\tilde{\boldsymbol{v}} \in \boldsymbol{X} \mid \boldsymbol{\nabla}.\tilde{\boldsymbol{v}} = 0\}$.

The weak formulation of (2.6) and (2.7) is: given $\boldsymbol{g} \in H^{-k}(\Omega)$ and, find $(\tilde{\boldsymbol{u}}, p) \in \boldsymbol{W} \times Q$ as

$$\begin{cases} a(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) + b(\tilde{\boldsymbol{v}},p) = \langle \boldsymbol{g},\tilde{\boldsymbol{v}}\rangle, \ \ \forall\, \tilde{\boldsymbol{v}} \in \boldsymbol{W} \\ b(\tilde{\boldsymbol{u}},q) = 0, \ \ \forall\, q \in Q \end{cases} \tag{2.8}$$

Where the bilinear form $a\colon \boldsymbol{W} \times \boldsymbol{W} \to \mathbb{R}$ is define by:

$$a(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) = \frac{\alpha}{N.\Delta t}(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) + \frac{1}{M^2}(\boldsymbol{\nabla}.\tilde{\boldsymbol{u}},\boldsymbol{\nabla}.\tilde{\boldsymbol{v}}) + (\boldsymbol{\nabla}\phi - \boldsymbol{u}\times\boldsymbol{B},\boldsymbol{\nabla}\psi - \boldsymbol{v}\times\boldsymbol{B})$$

And the bilinear for $b\colon \boldsymbol{W} \times q \to \mathbb{R}$ is:

$$b(\tilde{\boldsymbol{v}},q) = -\int_\Omega q\boldsymbol{\nabla}.\tilde{\boldsymbol{u}}$$

In the practice $\boldsymbol{g} = \boldsymbol{f} + \boldsymbol{u}^n o\boldsymbol{X}^n$.

We can prove the existence of unique solution of the problem (2.8) from the Lax – Milgram theorem.

In addition the equations (2.8) are analogous to optimum conditions of a quadratic functional minimization problem. The pressure appear as a lagrangian multiply if we consider the zero divergence $\boldsymbol{\nabla}.\tilde{\boldsymbol{u}} = \boldsymbol{0}$ as a linear constraint on the solution of $\tilde{\boldsymbol{u}}$. In fact, we can define a functional $J(\tilde{\boldsymbol{v}})$ got from (2.8) by:

$$J(\tilde{\boldsymbol{v}}) = \frac{1}{2}\,a(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}}) - \langle \boldsymbol{g},\tilde{\boldsymbol{v}}\rangle \tag{2.9}$$

The problem (2.9) becomes a problem of minimization of $J(\tilde{\boldsymbol{v}})$ under the constraint $\tilde{\boldsymbol{u}}$ with $\tilde{\boldsymbol{u}} \in V$

$$\begin{cases} J(\tilde{\boldsymbol{u}}) \le J(\tilde{\boldsymbol{v}}), \qquad \forall\, \tilde{\boldsymbol{v}} \in V \\ \tilde{\boldsymbol{u}} \in V \end{cases} \tag{2.10}$$

To overcome the difficulties resulting from the constraint, we can transform (2.10) into a saddle point problem. For it, we define, for $\tilde{\boldsymbol{v}} \in \boldsymbol{W}$ and $q \in L^2(\Omega)$ the lagrangian:

$$\mathcal{L}(\tilde{\boldsymbol{v}},q) = J(\tilde{\boldsymbol{v}}) - (q,\boldsymbol{\nabla}.\tilde{\boldsymbol{v}}) \tag{2.11}$$

The problem (2.10) comes down to the roughness of the couple $(\tilde{\boldsymbol{u}},p)$, a saddle – point of $\mathcal{L}$ on $\boldsymbol{W} \times L^2(\Omega)$, in other words, a solution of the problem

$$\begin{cases} \mathcal{L}(\tilde{\boldsymbol{u}},q) \le \mathcal{L}(\tilde{\boldsymbol{u}},p) \le \mathcal{L}(\tilde{\boldsymbol{v}},p), \qquad \forall\, (\tilde{\boldsymbol{v}},q) \in \boldsymbol{W} \times L^2(\Omega) \\ \tilde{\boldsymbol{u}} \in \boldsymbol{W}, p \in L^2(\Omega) \end{cases} \tag{2.12}$$

1.4. Uzawa method

For determining the saddle point, we used the Uzawa algorithm (See [1], [2], [3]):

Given a parameter $\alpha > 0$, called a relaxation parameter, the Uzawa algorithm for approximating $(\tilde{\boldsymbol{u}},p)$ of (2.12) can be describe as follows:

Step 1: Choose arbitrary p^0

Step 2: for $k \ge 0$, p^k is known , compute $\tilde{\boldsymbol{u}}^{k+1}$ by :

$$\mathcal{L}(\widetilde{\boldsymbol{u}}^{\boldsymbol{k+1}}, p^{k+1}) \leq \mathcal{L}(\widetilde{\boldsymbol{v}}, p^k), \qquad \forall\, \widetilde{\boldsymbol{v}} \in \boldsymbol{W}, \widetilde{\boldsymbol{u}} \in \boldsymbol{W}$$

Step 3: then p^{k+1} is deduced by

$$p^{k+1} = p^k + \alpha \boldsymbol{\nabla}.\widetilde{\boldsymbol{u}}^{\boldsymbol{k+1}}$$

Step 4: Stop when

$$\|p^{k+1} - p^k\| \leq \varepsilon$$

1.5. Finite element approximation

In this work, we choose the mixed formulation due to difficulty of discretize space V. We fix $h > 0$. Let T_h be a triangulation of $\overline{\Omega} \subset \mathbb{R}^d$ such as $\overline{\Omega} = \cup_{K \in T_h} K$ and $diameter(K) \leq h$ and two any closed element are either disjoint. We define finite dimensional spaces $\boldsymbol{X_h} \subset \boldsymbol{X}$, $Y_h \subset Y$, $Q_h \subset Q$ and $\boldsymbol{W_h} = \boldsymbol{X_h} \times Y_h$. We assume that $\boldsymbol{X_h} \times Q_h$ verify the discrete inf – sup condition to assure the stability of pressure. In addition, we suppose that $\boldsymbol{X_h}, Y_h, Q_h$ satisfy approximation properties of piecewise polynomial and we can proved that the best choice for a space Y_h is to take the same order element that $\boldsymbol{X_h}$. Thus, we choose the classical (P_2 / P_1) Taylor – Hood element for velocity and pressure and P_2 for electric potential. The mixed formulation of variationnelle problem (2.8) is written:

For $\widetilde{\boldsymbol{v}}_{\boldsymbol{h}} = (\boldsymbol{v_h}, \psi_h) \in \boldsymbol{W_h}$ and $q_h \in Q_h$, Find $\widetilde{\boldsymbol{u}}_{\boldsymbol{h}} = (\boldsymbol{u_h}, \phi_h) \in \boldsymbol{W_h}$ and $p_h \in Q_h$ such as

$$\begin{cases} a_h(\widetilde{\boldsymbol{u}}_{\boldsymbol{h}}, \widetilde{\boldsymbol{v}}_{\boldsymbol{h}}) + b_h(\widetilde{\boldsymbol{v}}_{\boldsymbol{h}}, p_h) = \langle \boldsymbol{g_h}, \widetilde{\boldsymbol{v}} \rangle, \ \ \forall\, \widetilde{\boldsymbol{v}}_{\boldsymbol{h}} \in \boldsymbol{W_h} \\ b_h(\widetilde{\boldsymbol{u}}_{\boldsymbol{h}}, q_h) = 0, \ \ \forall\, q_h \in Q_h \end{cases} \tag{2.13}$$

III. Results et discussions

In this section, we present the comparison between the numerical results obtained by our method and exact solutions resulting from the literature. And we compute the rate convergence in order to verify the convergence and accuracy of our approaches. The code of our simulation are realized by using the software FreeFem ++.

For that, we take $\Omega = [0,1] \times [0,1] \times [0,1]$, Re = 1600, M = 200, N = 25 and the external magnetic $\boldsymbol{B} = (0,0,1)$. The body force $\boldsymbol{f}$, boundary and initial condition are determined by the exact solution. We consider the exact solution $(\boldsymbol{u}, p, \phi)$ described in [6] and [7] :

$$\boldsymbol{u}(x,y,t) = (2\pi cos(2\pi x)sin(2\pi y), -2\pi sin(2\pi x)cos(2\pi y), 0)e^{-5t},$$
$$p(x,y,t) = 0,$$
$$\phi(x,y,t) = (cos(2\pi x)cos(2\pi y) + x^2 + y^2)e^{-5t}$$

1.6. Numerical result

Figure 1 and figure 2 represent the confrontation between the approximated solution and the exact solution for velocity.

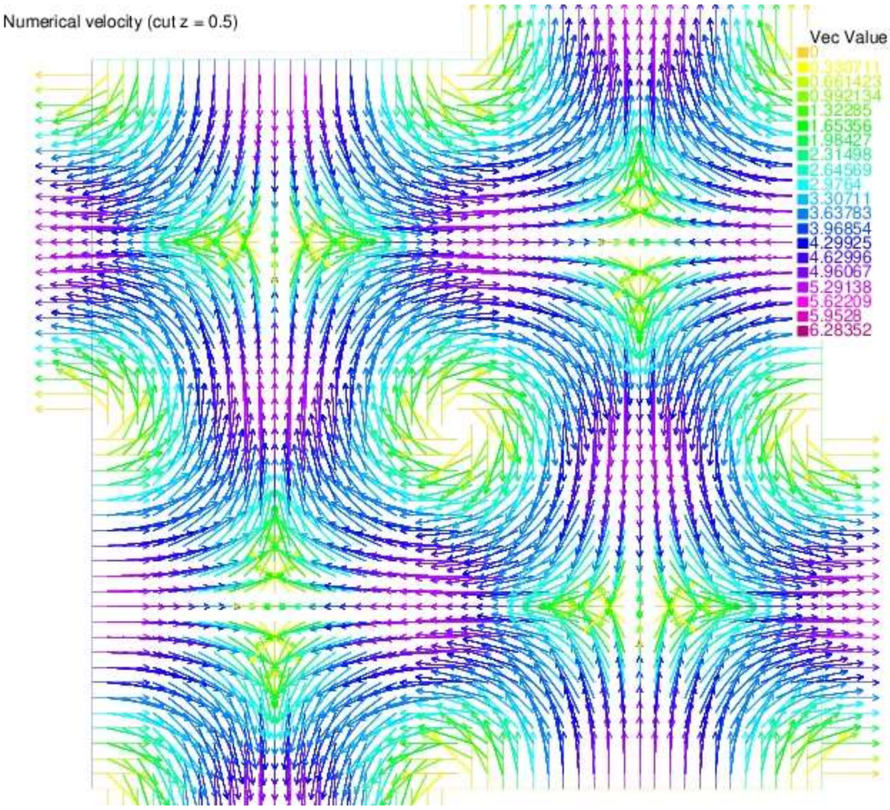

Figure 1 Computed solution of velocity on unite cube

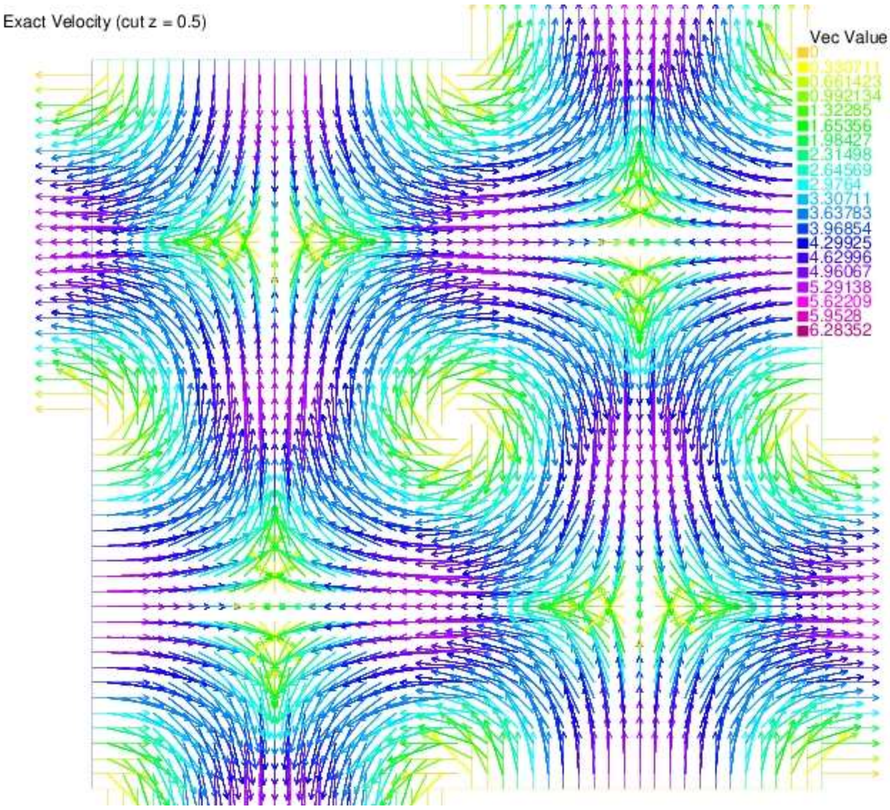

Figure 2 Exact solution of velocity on the unit cube

Figure 3 and figure 4 allow to compare the numerical and analytical solutions relating to the electric potential.

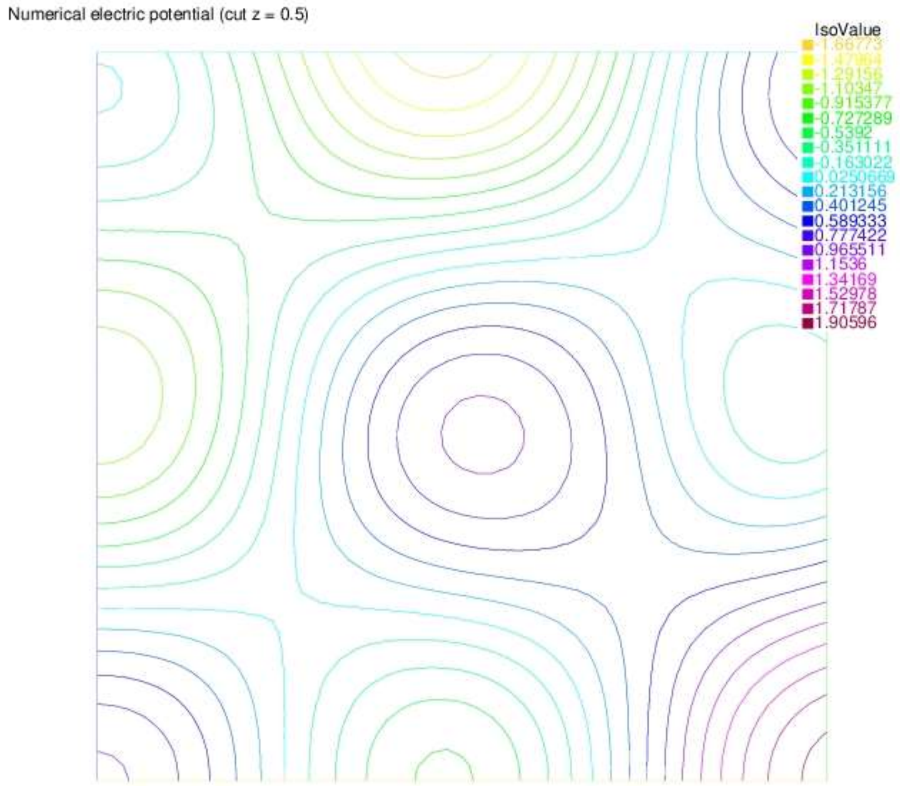

Figure 3 Computed solution of electric potential on the unit cube

Figure 4 Exact solution of electric potential on the unit cube

These results show that the approximated solutions and the exact solutions are similar.

1.7. Rate of convergence

To make sure of the similarity, we calculate the space errors on the velocity and electric potential in the Lebesgue norm and Sobolev norm:

$$\|\boldsymbol{u}-\boldsymbol{u_h}\|_0=\left(\int_\Omega \sum_{i=1}^{3}(u_i-u_{i,h})^2\right)^{\frac{1}{2}}$$

$$\|\boldsymbol{u}-\boldsymbol{u_h}\|_1=\left(\int_\Omega \|u_i-u_{i,h}\|_{L^2(\Omega)}+\sum_{j=1}^{3}\left\|\frac{\partial u_i}{\partial x_j}-\frac{\partial u_{i,h}}{\partial x_j}\right\|_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

$$\|\phi-\phi_h\|_0=\left(\int_\Omega (\phi-\phi_h)^2\right)^{\frac{1}{2}}$$

$$\|\phi-\phi_h\|_1=\left(\int_\Omega \|\phi-\phi_h\|_{L^2(\Omega)}+\sum_{j=1}^{3}\left\|\frac{\partial \phi}{\partial x_j}-\frac{\partial \phi_h}{\partial x_j}\right\|_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

We resume these error in the table 1.

Table 1: Convergence performance of Uzawa algorithm

h	$\|\boldsymbol{u}-\boldsymbol{u_e}\|_0$	Rate	$\|\boldsymbol{u}-\boldsymbol{u_e}\|_1$	Rate	$\|\phi-\phi_e\|_0$	Rate	$\|\phi-\phi_e\|_1$	Rate
0.866025	0.159415	-	0.233701	-	0.19981	-	0.336425	-
0.433013	0.00996845	3.999	0.0330067	2.824	0.0336953	2.568	0.07573	2.151
0.216506	6.30446e-4	3.982	0.00353278	3.224	0.00306644	3.458	0.0140951	2.426
0.108253	4.06642e-5	3.955	3.97622e-4	3.151	2.17067e-4	3.820	0.00205706	2.777

We show in figure 5 and 6 the convergence curves for velocity an electric potential.

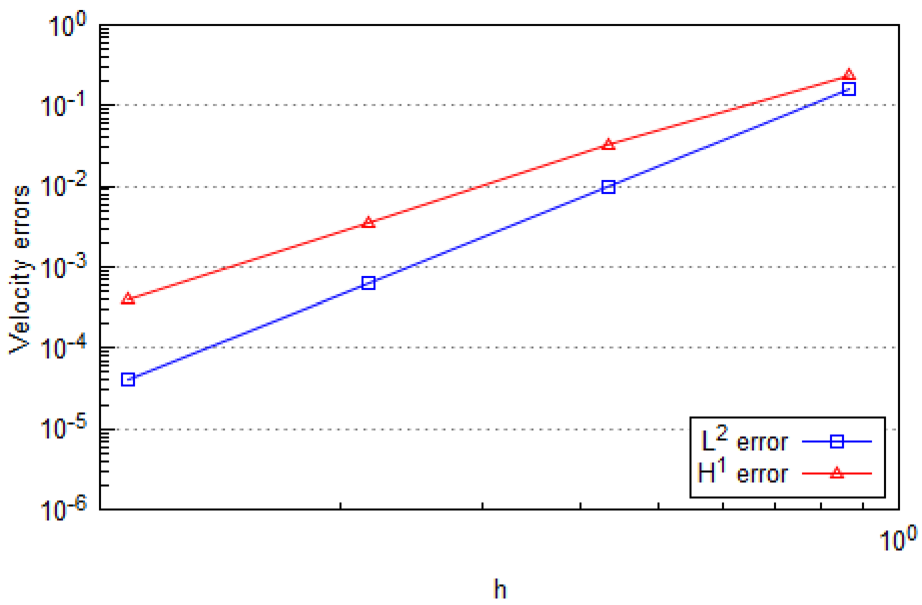

Figure 5 convergence curves (logarithmic scale) for velocity in $L^2(\Omega)$ and $H^1(\Omega)$

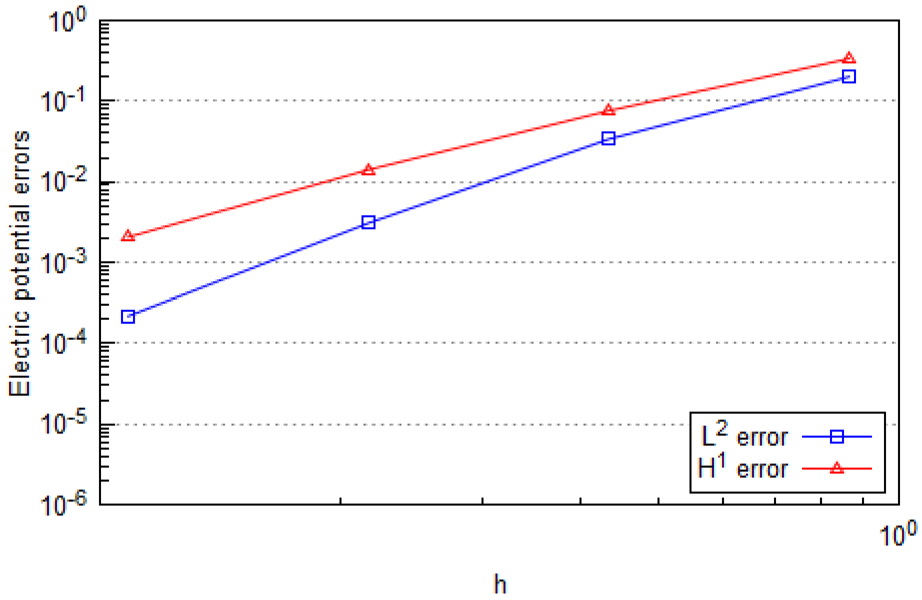

Figure 6 convergence curves (logarithmic scale) for electric potential in $L^2(\Omega)$ and $H^1(\Omega)$

We can deduce from these results that the errors decrease when the size of the mesh tends towards zero. In addition, we see the rate of convergence in $L^2(\Omega)$ and $H^1(\Omega)$ for respectively velocity and electric potential converge to four and three order. This proves that our method is good.

IV. Conclusions

To conclude, the important applications in the field of industry and engineering of magnetohydrodynamic lead us to propose a numerical solution of the equations of low Reynolds number magnetohydrodynamic. We started to apply the characteristic method in order to absorb the nonlinearity. We use the fully coupled time discretization and semi – implicit Euler scheme for stability. This lead us to solve a succession of continuous problems in the Hilbert space. The problem is similar to the minimization problem under a constraint of a quadratic functional. It follows that resolution is reduced to search a saddle point of a lagrangian. We choose the classical Uzawa algorithm to seek the saddle point of the lagrangian. We realized a comparison between the approximate solutions with analytical solutions presented in the literature. In addition, we computed the space errors on the velocity and electric potential in the Lebesgue and Sobolev norm and their rates of convergence. The results showed that the numerical solutions obtained by our approaches similar to the exact solutions. We observed the decreases of errors when the meshes size toward to zero with a rate convergence four and three in respectively Lebesgue and Sobolev space for velocity and electric potential. This proves that our method converge with good accuracy.

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