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Derivation and Implementation of a New Fourth-Stage Second-Order Runge-Kutta Formula Using Maple Program

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ABSTRACT: *This paper is aimed at deriving a new fourth-stage second-order Runge-Kutta formula and to implement it using MAPLE program on initial value problems in Ordinary Differential Equations. The intention is to find out whether this order can improve the performance of results when we increase the number of stages and vary the parameters properly.*

Keywords: *Fourth-stage, Second-Order, Ordinary Differential Equations, Runge-Kutta methods.*

I. INTRODUCTION

The objective of this work is to use Runge-Kutta Methods to solve initial value problems in Ordinary Differential Equations of the form $y^1 = f(x, y)$, $y(x_0) = y_0$ $a \le x \le b$, with a view to finding out whether an increase in the stage of a second-order method can improve the performance of the method.

In the work of Agbeboh (2013), the Runge-Kutta Methods represent an important family of implicit and explicit iterative methods for the approximation of ordinary differential equations in numerical analysis. Furthermore, of all computational methods for the numerical solution of initial-value problems, the easiest to implement is Euler's rule. It is explicit and, being a one-step method, requires no additional starting values and readily permits a change of step length during the computation. Its low order, of course, makes it of limited practical value. According to Butcher (1975, 1964, 1988), linear multi-step methods achieve higher order by sacrificing the one-step nature of the algorithm, while retaining linearity with respect to y_{n+j}, f_{n+j}, $j = 0, 1 \ldots k$. Higher order can also be achieved by sacrificing linearity, but preserving the one-step nature of the algorithm.

Thus, according to Agbeboh, Ukpebor and Esekhaigbe (2009), "the philosophy behind the Runge-Kutta method is to retain the advantages of one-step methods". Due to the loss of linearity, error analysis is considerably more difficult than in the case of linear multi-step methods as said by William (2002). According to Butcher (2000, 2009) and Vander Houwen (2014, 2015), Runge-Kutta methods are all explicit, although, recently, implicit Runge-Kutta Methods, which have improved weak stability characteristics, have been considered.

1.1 Objectives of the study:

i. To analyze the coefficients of the fourth-stage second-order Runge-Kutta Method using Taylor's series expansion.

ii. To solve initial-value problems using the fourth-stage second-order Runge-Kutta formulae.

iii. To develop MAPLE programs based on this formula for solving initial-value problems.

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iv. To see if the results from the fourth-stage second-order formula has relatively minimal error than that of the second-stage second-order method.

II. Methodology

We adopted the methods below in carrying out our research:

i. We analyzed and expanded the $k_{i,s}$ of the fourth-stage second-order Runge-Kutta method using Taylor series expansion about the point (x,y).

ii. We compared our results with the results of the Taylor series expansion and equated coefficients.

iii. We varied parameters so as to enable us satisfy the set of non-linear equations with the aim of getting a new fourth-stage second- order Runge-Kutta formula.

iv. We implemented the formula in solving initial-value problems with the aid of MAPLE programs.

v. We compared the results with the results from the existing second-stage second-order formula to see the level of performance of our new method.

2.1 THE GENERAL SCHEME OF THE EXPLICIT RUNGE-KUTTA METHODS

According to Butcher (2009), the general R-stage Runge-Kutta method is:

$$
\begin{aligned}
y_{n+1} - y_n &= h\,\phi(x_n, y_n; h), \\
\phi(x_n, y_n; h) &= \sum_{r=1}^{R} c_r k_r, \\
k_1 &= f(x,y), \qquad\qquad (2.1) \\
k_r &= f\Big(x + h a_r,\ y + h\sum_{s=1}^{r-1} b_{rs} k_s\Big), \quad r = 2, 3, \ldots R, \\
a_r &= \sum_{s=1}^{r-1} b_{rs}, \qquad r = 2, 3, \ldots, R.
\end{aligned}
$$

Now, the Taylor's series gives:

$$y(x_{n+1}) = y(x_n) + hy^{1}(x_n) + h^2/_{2!}\, y^{11}(x_n) + h^3/_{3!}\, y^{111}(x_n) + \ldots \qquad (2.2)$$

where,

$$
\begin{aligned}
y^{1} &= f(x, y) \\
y^{11} &= f^{1} = f_x + f_y y^{1} = f_x + f_y f \\
y^{111} &= f^{11} = f_{xx} + f_{xy}f + f_{yx}f + f_{yy}f^2 + f_y f_x + f_y^{\,2} f \\
&= f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^{\,2} f \\
y^{(iv)} &= f^{111} = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + 3f_x f_{xy} + 5ff_{xy}fy + 3ff_x f_{yy} + f_{xx}f_y + 4f^2 f_y f_{yy} + f_x f_y^{\,2} + ff_y^{\,3} + f^3 f_{yyy} \\
y^{(v)} &= f^{iv} = f_{xxxx} + 4ff_{xxxy} + 6f^2 f_{xxyy} + 6f_x f_{xxy} + 9ff_{xxy}fy + 4f^3 f_x f_{yyy} + 6f_{xx}f_{xyy} + 15f^2 f_{xyy}f_y + 6ff_x f_{xyy} + 4f_{xx}f_{xy} + \\
& 8ff^2 f_{xy} + 8f^2 f_{xy} f_{yy} + 7f_x f_y + 9ff_{xy}f_y^{\,2} + 6f^2 f_x f_{yyy} + 4ff_{xx}f_{yy} + 3f_x^{\,2} f_{yy} + 13ff_x f_y f_{yy} + f_{xxx}f_y + 7f^3 f_y f_{yyy} + 4f^3 f_{yy}^{\,2} + 4f^2 f_{xy} f_{yy} \\
& + 11f^2 f_y^{\,2} f_{yy} + f_{xx}f_y^{\,2} + f_x f_y^{\,2} + f_x f_y^{\,3} + ff_y^{\,4} + f^4 f_{yyyy},
\end{aligned}
$$

where f_x, f_y represent the derivatives of f with respect to x,y respectively.

III. THE SECOND-ORDER FOURTH-STAGE EXPLICIT RUNGE-KUTTA METHOD

The derivation of the second-order fourth-stage Runge-Kutta formula is as follows: From the scheme above, we have:

$$
\begin{aligned}
y_i &= y_{i-1} + h_i(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4) \\
k_1 &= f(x_{i-1}, y_{i-1}) \\
k_2 &= f(x_{i-1} + c_2 h_i, y_{i-1} + h_i(a_{21}k_1)) \\
k_3 &= f(x_{i-1} + c_3 h_i, y_{i-1} + h_i(a_{31}k_1 + a_{32}k_2)) \\
k_4 &= f(x_{i-1} + c_4 h_i, y_{i-1} + h_i(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)) \qquad (3.1)
\end{aligned}
$$

Using Taylor's Series Expansion for $k_{i's}$ $h_i = h$

$$k_1 = f(x_{i-1}, y_{i-1}) = f$$

$k_2 = \sum_{r=0}^{\infty} \frac{1}{r!} (c_2 h \frac{\delta}{\delta x} + h a_{21} k_1 \frac{\delta}{\delta y})^r f$

$k_3 = \sum_{r=0}^{\infty} \frac{1}{r!} (c_3 h \frac{\delta}{\delta x} + h(a_{31} k_1 + a_{32} k_2) \frac{\delta}{\delta y})^r f$

$k_4 = \sum_{r=0}^{\infty} \frac{1}{r!} (c_4 h \frac{\delta}{\delta x} + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3) \frac{\delta}{\delta y})^r f$ (3.2)

Expanding we have:

$k_1 = f$

$k_2 = f + (c_2 h f_x + h a_{21} k_1 f_y) + \frac{1}{2!}(c_2 h f_x + h a_{21} k_1 f_y)^2 + \frac{1}{3!}(c_2 h f_x + h a_{21} k_1 f_y)^3 + \frac{1}{4!}(c_2 h f_x + h a_{21} k_1 f_y)^4 + (0h^5)$

$k_3 = f + (c_3 h f_x + h(a_{31} k_1 + a_{32} k_2) f_y) + \frac{1}{2!}(c_3 h f_x + h(a_{31} k_1 + a_{32} k_2) f_y)^2 + + \frac{1}{3!}(c_3 h f_x + h(a_{31} k_1 + a_{32} k_2) f_y)^3 + \frac{1}{4!}(c_3 h f_x + h(a_{31} k_1 + a_{32} k_2) f_y)^4 + (0h^5)$

$k_4 = f + (c_4 h f_x + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3) f_y) + \frac{1}{2!}(c_4 h f_x + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3) f_y)^2 + \frac{1}{3!}(c_4 h f_x + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3) f_y)^3 + \frac{1}{4!}(c_4 h f_x + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3) f_y)^4 + (0h^5)$. (3.3)

Now expanding the $k_{i's}$ in terms of f only:

$k_1 = f$

$k_2 = f + h(c_2 f_x + a_{21} f f_y) + \frac{h^2}{2!}(c_2^2 f_{xx} + 2c_2 a_{21} f f_{xy} + a_{21}^2 f^2 f_{yy}) + 0(h^3)$

$k_3 = f + h(c_3 f_x + (a_{31} + a_{32}) f f_y) + \frac{h^2}{2!}(c_3^2 f_{xx} + 2c_3(a_{31} + a_{32}) f f_{xy} + (a_{31}^2 + 2a_{31} a_{32} + a_{32}^2) f^2 f_{yy} + 2c_2 a_{32} f_x f_y + 2a_{31} a_{32} f f_y^2) + 0(h^3)$

$k_4 = f + h(c_4 f_x + (a_{41} + a_{42} + a_{43}) f f_y) + \frac{h^2}{2!}(c_4^2 f_{xx} + (2c_2 a_{42} + 2c_3 a_{43}) f_x f_y + (2c_4 a_{41} + 2c_4 a_{42} + 2c_4 a_{43}) f f_{xy} + (2a_{21} a_{42} + a_{31} a_{43} + 2a_{32} a_{43}) f f_y^2 + (2a_{41}^2 + 4a_{41} a_{42} + 4a_{41} a_{43} + 2a_{42}^2 + 4a_{42} a_{43} + 2a_{43}^2) f^2 f_{yy}) + 0(h^3)$ (3.4)

But $\emptyset(x, y, h) = \sum_{r=1}^{4} b_r k_r = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4)$

$\emptyset(x, y, h) = (b_1 f + b_2 \left(f + h(c_2 f_x + a_{21} f f_y) + \frac{h^2}{2!}(c_2^2 f_{xx} + 2c_2 a_{21} f f_{xy} + a_{21}^2 f^2 f_{yy}) \right) + b_3(f + h(c_3 f_x + (a_{31} + a_{32}) f f_y) + \frac{h^2}{2!}(c_3^2 f_{xx} + 2c_3(a_{31} + a_{32}) f f_{xy} + (a_{31}^2 + 2a_{31} a_{32} + a_{32}^2) f^2 f_{yy} + 2c_2 a_{32} f_x f_y + 2a_{31} a_{32} f f_y^2)) + b_4(f + h(c_4 f_x + (a_{41} + a_{42} + a_{43}) f f_y) + \frac{h^2}{2!}(c_4^2 f_{xx} + (2c_2 a_{42} + 2c_3 a_{43}) f_x f_y + (2c_4 a_{41} + 2c_4 a_{42} + 2c_4 a_{43}) f f_{xy} + (2a_{21} a_{42} + a_{31} a_{43} + 2a_{32} a_{43}) f f_y^2 + (2a_{41}^2 + 4a_{41} a_{42} + 4a_{41} a_{43} + 2a_{42}^2 + 4a_{42} a_{43} + 2a_{43}^2) f^2 f_{yy})))$

(3.5)

The Taylor Series Expansion is:

$\emptyset_T(x, y, h) = f + \frac{1}{2} h(f_x + f f_y) + \frac{1}{6} h(f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2) + 0(h^3)$ (3.6)

Equating (3.5) with (3.6), we have the Equations below:

$b_1 + b_2 + b_3 + b_4 = 1$ (3.7)

$b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{2}$ (3.8)

$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = \frac{1}{2}$ (3.9)

$b_2 c_2^2 + b_3 c_3^2 + b_4 c_4^2 = \frac{1}{3}$ (3.10)

$b_2 c_2 a_{21} + b_3 c_3(a_{31} + a_{32}) + b_4 c_4(a_{41} + a_{42} + a_{43}) = \frac{2}{3}$ (3.11)

$b_2 c_{21}^2 + b_3(a_{31} + a_{32})^2 + 2b_4(a_{41} + a_{42} + a_{43})^2 = \frac{1}{3}$ (3.12)

$2b_3 c_2 a_{32} + 2b_4 c_2 a_{42} + 2b_4 c_3 a_{43} = \frac{1}{3}$ (3.13)

$2b_3 a_{21} a_{32} + 2b_4 a_{21} a_{42} + 2b_4 a_{43}(a_{31} + a_{32}) = \frac{1}{3}$ (3.14)

Solving the above Eight (8) equations (3.7 to 3.14), we set

$c_1 = 0, \quad c_4 = 1, c_2 = \frac{1}{4}, \quad c_3 = \frac{3}{4},$ (3.15)

The eight (8) Equations become:

$b_1 + b_2 + b_3 + b_4 = 1$

$b_2 + 3b_3 + 4b_4 = 2$ (3.16)

$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = \frac{1}{2}$ (3.17)

$b_2 a_{21} + 3b_3(a_{31} + a_{32}) + 4b_4(a_{41} + a_{42} + a_{43}) = \frac{8}{3}$ (3.18)

$b_2 + 9b_3 + 16b_4 = \frac{16}{3}$ (3.19)

$b_2 a_{21}^2 + b_3(a_{31} + a_{32})^2 + 2b_4(a_{41} + a_{42} + a_{43})^2 = \frac{1}{3}$ (3.20)

$b_3 a_{32} + b_4 a_{42} + 3b_4 a_{42} = \frac{2}{3}$ (3.21)

$b_3 a_{31} a_{32} + b_4 a_{21} a_{42} + b_4 a_{43}(a_{31} a_{32}) = \frac{1}{6}$ (3.22)

Solving (3.7), (3.16) and (3.19), we have:

$b_1 = \frac{1}{4}, \quad b_2 = \frac{1}{18}, \quad b_3 = \frac{5}{6}, \quad b_4 = -\frac{5}{36}$ (3.23)

Letting $a_{21} = A, a_{31} + a_{32} = B, \quad a_{41} + a_{42} + a_{43} = C$ (3.24)

Putting (3.15), (3.23) and (3.24) into the remaining equations, we have:

$a_{21} = \frac{3}{28}, \quad a_{31} = -\frac{283}{140}, \quad a_{32} = \frac{6}{5}, \quad a_{41} = -\frac{339}{35}, \quad a_{42} = \frac{3}{5}, \quad a_{43} = \frac{3}{5}$ (3.25)

Putting (3.15), (3.23) and (3.125) into (3.1), we have our new Second Order Fourth-Stage formula below:

$$y_i = y_{i-1} + \frac{h_i}{36}(9k_1 + 2k_2 + 30k_3 - 5k_4),$$
$$k_1 = f(x_{i-1}, y_{i-1}),$$
$$k_2 = f\left(x_{i-1} + \frac{h_i}{4}, \quad y_{i-1} + \frac{3h_i}{28}k_1\right),$$
$$k_3 = f\left(x_{i-1} + \frac{3h_i}{4}, y_{i-1} + \frac{h_i}{140}(-283k_1 + 168k_2)\right),$$
$$k_4 = f\left(x_{i-1} + h_i, y_{i-1} + \frac{h_i}{35}(-339k_1 + 21k_2 + 21k_3)\right).$$

The Existing Second-Stage Second-Order Method is:

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2),$$
$$k_1 = f(x_n, y_n),$$
$$k_2 = f(x_n + h, y_n + hk_1)$$

IV. NUMERICAL COMPUTATIONS AND RESULTS

In order to access the performance of our new fourth-stage second-order Runge-Kutta methods, the following sample problems were solved:

(i) $y' = y, y(0) = 1.0, \ 0 \leq x \leq 1, \ y(x_n) = e^x, h = 0.1$

(ii) $y' = -y, \ y(0) = 1.0, \ 0 \leq x \leq 1, \ y(x_n) = e^{-x}, h = 0.1$

4.1 TABLES OF RESULTS

Fourth-stage Second-order (Problem 1)

XN	YN	TSOL	ERROR
.1D+00	1.10516559523809523810	1.10517091807564762480	0.00000532283755238670
.2D+00	1.22139099289797335600	1.22140275816016983390	0.00001176526219647790
.3D+00	1.34983930368453687750	1.34985880757600310400	0.00001950389146622650
.4D+00	1.49179595753229720090	1.49182469764127031780	0.00002874010897311690
.5D+00	1.64868156737996548140	1.64872127070012814680	0.00003970332016266540

Second-stage Second-order (Problem 1)

XN YN TSOL ERROR

.1D+00	1.10500000000000000000	1.10517091807564762480	0.00017091807564762480
.2D+00	1.22102500000000000000	1.22140275816016983390	0.00037775816016983390
.3D+00	1.34923262500000000000	1.34985880757600310400	0.00062618257600310400
.4D+00	1.49090205062500000000	1.49182469764127031780	0.00092264701627031780
.5D+00	1.64744676594062500000	1.64872127070012814680	0.00127450475950314680

Fourth-stage Second-order (Problem 2)

XN	YN	TSOL	ERROR
.1D+00	0.90483226190476190476	0.90483741803595957316	0.00000515613119766840
.2D+00	0.81872142218368764172	0.81873075307798185867	0.00000933089429421695
.3D+00	0.74080555630434959958	0.74081822068171786607	0.00001266437736826649
.4D+00	0.67030476714248009850	0.67032004603563930074	0.00001527889315920224
.5D+00	0.60651337861907499457	0.60653065971263342360	0.00001728109355842903

Second-stage Second-order (Problem 2)

XN	YN	TSOL	ERROR
.1D+00	0.90500000000000000000	0.90483741803595957316	−0.00016258196404042684
.2D+00	0.81902500000000000000	0.81873075307798185867	−0.00029424692201814133
.3D+00	0.74121762500000000000	0.74081822068171786607	−0.00039940431828213393
.4D+00	0.67080195062500000000	0.67032004603563930074	−0.00048190458936069926
.5D+00	0.60707576531562500000	0.60653065971263342360	−0.00054510560299157640

V. FINDINGS AND CONCLUSION

After the implementation, we discovered that the method with the higher stage which happens to be our method has a better accuracy even when the two methods have order two. This can also be seen in our tables of results. Hence, our research has revealed that when we increase the stage of a Runge-Kutta method and vary the parameters properly, the accuracy and performance of such a method is bound to increase.

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